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Bruhat order on the involutions of classical Weyl groups

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Abstract

It is well known that a Coxeter group W, partially ordered by the Bruhat order, is a graded poset, with rank function given by the length, and that it is *EL*-shellable, hence Cohen–Macaulay, and Eulerian. We ask whether $Invol(W)$, the subposet of W induced by the set of involutions, is endowed with similar properties. If W is of type A or B, we proved, respectively in [F. Incitti, The Bruhat order on the involutions of the symmetric group, J. Algebraic Combin. 20 (2004), 243–261] and [F. Incitti, The Bruhat order on the involutions of the hyperoctahedral group, European J. Combin. 24 (2003), 825–848], that $Invol(W)$ is graded, *EL*-shellable and Eulerian. In this work we complete the investigation on the classical Weyl groups, extending these results to type D and providing a unified description for the rank function.

MSC: 05E15; 20F55

Keywords: Coxeter groups; Classical Weyl groups; Bruhat order; Involutions; *EL*-shellability

1. Introduction

It is known that a Coxeter group W, partially ordered by the Bruhat order, is a graded poset, with rank function given by the length, and that it is also *EL*-shellable, hence Cohen–Macaulay, and Eulerian. We ask whether a particular subposet of W, namely that induced by the set of involutions of W, which we denote by $Invol(W)$, is endowed with similar properties.

The problem arises from a geometric question. It is known that a classical Weyl group W, partially ordered by the Bruhat order, encodes the cell decomposition of a Schubert variety. In [20,21] Richardson and Springer introduced a vast generalization of this partial order, in relation

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to the cell decomposition of certain symmetric varieties. In a particular case they obtained the subposet of W induced by the involutions.

In [13,14] we proved that if W is of type A or B, respectively, then $Invol(W)$ is graded, *EL*-shellable and Eulerian. In this work we complete the investigation on the classical Weyl groups, extending these results to type D and providing a unified description for the rank function, which turns out to be the average between the length and the absolute length. In order to prove the result for the even-signed permutation group, it has been crucial to find a unified approach for the three families of classical Weyl groups.

In [16] we also conjectured that our results were valid for every Coxeter group. A partial answer has been recently given by Hultman [11], who proved some of these results, including the description of the rank function, for general Coxeter groups, using completely different techniques.

The organization of this paper is as follows. Some basic definitions and results are collected in Section 2. In Section 3 we find a combinatorial description of the absolute length of involutions in classical Weyl groups. The main result is stated in Section 4. Section 5 is devoted to the exposition of the results about the hyperoctahedral group, as they are needed later. Finally, in Section 6 we prove the results for the even-signed permutation group.

2. Notation and preliminaries

We let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{Z}$ be the set of integers. For $n, m \in \mathbf{Z}$, with $n \leqslant m$, we let $[n, m] = \{n, n+1, \ldots, m\}$. For $n \in \mathbf{N}$, we let $[n] = [1, n]$, $[-n] = [-n, -1]$ and $[\pm n] = [-n, n] \backslash \{0\}$. We denote by $\equiv$ the congruence modulo 2: $n \equiv m$, with $n, m \in \mathbf{Z}$, means that $n - m$ is even. Finally, we denote simply by $<$ the lexicographic order between n-tuples: $(a_1, a_2, \ldots, a_n) < (b_1, b_2, \ldots, b_n)$ means that $a_k < b_k$, where $k = \min\{i \in [n]\colon a_i \neq b_i\}$.

2.1. Posets

We follow [22, Chapter 3] for poset notation and terminology. In particular we denote by $\lhd$ the *covering relation*: $x \lhd y$ means that $x < y$ and there is no z such that $x < z < y$. A poset is *bounded* if it has a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$ respectively. If $x, y \in P$, with $x \leqslant y$, we let $[x, y] = \{z \in P\colon x \leqslant z \leqslant y\}$, and we call it an *interval* of P. If $x, y \in P$, with $x < y$, a *chain* from x to y of *length* k is a $(k+1)$ tuple $(x_0, x_1, \ldots, x_k)$ such that $x = x_0 < x_1 < \cdots < x_k = y$. A chain $x_0 < x_1 < \cdots < x_k$ is said to be *saturated* if all the relations in it are covering relations $(x_0 \lhd x_1 \lhd \cdots \lhd x_k)$.

A poset is said to be *graded* of *rank* n if it is finite, bounded and if all maximal chains of P have the same length n. If P is a graded poset of rank n, then there is a unique *rank function* $\rho : P \to [0, n]$ such that $\rho(\hat{0}) = 0$, $\rho(\hat{1}) = n$ and $\rho(y) = \rho(x) + 1$ whenever y covers x in P. Conversely, if P is finite and bounded, and if such a function exists, then P is graded of rank n.

Let P be a graded poset and let Q be a totally ordered set. An *EL-labeling* of P is a function $\lambda : \{(x, y) \in P^2\colon x \lhd y\} \to Q$ such that for every $x, y \in P$, with $x < y$, two properties hold:

(1) there is exactly one saturated chain from x to y with non-decreasing labels:

$$x = x_0 \underset{\lambda_1}{\lhd} x_1 \underset{\lambda_2}{\lhd} \cdots \underset{\lambda_k}{\lhd} x_k = y,$$

with $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k$;

(2) this chain has the lexicographically minimal labeling: if

$$x = y_0 \underset{\mu_1}{\lhd} y_1 \underset{\mu_2}{\lhd} \cdots \underset{\mu_k}{\lhd} y_k = y$$

is a saturated chain from x to y different from the previous one, then

$$(\lambda_1, \lambda_2, \ldots, \lambda_k) < (\mu_1, \mu_2, \ldots, \mu_k).$$

A graded poset P is said to be *EL-shellable* if it has an *EL*-labeling.

Connections between *EL*-shellable posets and shellable complexes, Cohen–Macaulay complexes and Cohen–Macaulay rings can be found, for example, in [1,3,4,9,10,19,23]. Here we only recall the following result, due to Björner.

Theorem 2.1. *Let P be a graded poset. If P is EL-shellable then P is shellable and hence Cohen–Macaulay.*

Finally, a graded poset P with rank function ρ is said to be *Eulerian* if

$$\left|\left\{z \in [x, y]\colon\ \rho(z) \text{ is even}\right\}\right| = \left|\left\{z \in [x, y]\colon\ \rho(z) \text{ is odd}\right\}\right|,$$

for every $x, y \in P$ such that $x < y$.

In an *EL*-shellable poset there is a necessary and sufficient condition for the poset to be Eulerian. We state it in the following form (see [4, Theorem 2.7] and [24, Theorem 1.2] for proofs of more general results).

Theorem 2.2. *Let P be a graded EL-shellable poset and let λ be an EL-labeling of P. Then P is Eulerian if and only if for every $x, y \in P$, with $x < y$, there is exactly one saturated chain from x to y with decreasing labels.*

We now recall some general techniques about posets, already introduced in [15].

Let P be a finite bounded poset. A *successor system* of P is a subset

$$H \subseteq \left\{(x, y) \in P^2\colon\ x < y\right\}.$$

An *insertion system* of P is a successor system H of P such that

(**insertion property**) for every $x, y \in P$, with $x < y$, there exists $z \in P$ such that $(x, z) \in H$ and $z \leqslant y$.

A *covering system* of P is a pair (H, ρ), where H is an insertion system of P and $\rho : P \to \mathbf{N} \cup \{0\}$ is a statistic on P such that

(ρ-**base property**) $\rho(\hat{0}) = 0$;
(ρ-**increasing property**) for every $(x, y) \in H$, we have $\rho(y) = \rho(x) + 1$.

If there exists a covering system (H, ρ) of P, then P is graded with rank function ρ (see [15, Proposition 3.2]). A covering system (H, ρ) also gives a complete description of the covering relation in P: the pairs of elements which are in covering relation are exactly the pairs in H (see [15, Proposition 3.3]).

Now let Q be a totally ordered set, the set of *labels*. Let H be a successor system of P. A *good labeling* of H is a function $\lambda : H \to Q$ such that

(**injectivity property**) for every $(x, y), (x, z) \in H$, we have

$$\lambda(x, y) = \lambda(x, z) \quad \Rightarrow \quad y = z.$$

Let H be a successor system of P and let λ be a good labeling of H. Let $x \in P$. An element $i \in Q$ is a *suitable label* of x if there is $y \in P$ such that $(x, y) \in H$ and $\lambda(x, y) = i$. By the injectivity property, such a y is unique, and we call it the *transformation* of x with respect to the label i, and denote it by

$$t_i^P(x).$$

The set of all suitable labels of x is denoted by $\Lambda(x)$.

The following is an equivalent version of the insertion property, once a good labeling of H is given:

(**insertion property**) *for every $x, y \in P$, with $x < y$, there exists a label $i \in \Lambda(x)$ such that $t_i^P(x) \leqslant y$.*

If (H, ρ) is a covering system of P, then we have $x \lhd y$ if and only if $(x, y) \in H$. In this case a good labeling λ of H is an edge-labeling of P. It is useful to introduce the following terminology: if $x \in P$ and $i \in \Lambda(x)$ then we call $t_i^P(x)$ the *covering transformation* of x with respect to the label i, and denote it by

$$ct_i^P(x).$$

Thus, for every $x \in P$, $i \in \Lambda(x)$ we have $x \lhd ct_i^P(x)$. On the other hand, if $x \lhd y$, then $y = ct_i^P(x)$ for a unique $i \in \Lambda(x)$, and we write also

$$x \underset{i}{\lhd} y.$$

We are now able to define the *minimal chains* in P. Note that, if (H, ρ) is a covering system of P, then by the insertion property, for every $x, y \in P$, with $x < y$, the set $\{i \in \Lambda(x)\colon\ ct_i^P(x) \leqslant y\}$ is not empty. This allows to give the following definition.

Definition 2.3. Let (H, ρ) be a covering system of P. Let $x, y \in P$, with $x < y$. The *minimal label* of x with respect to y, denoted by $mi_y(x)$ (or simply mi), is

$$mi_y(x) = \min\bigl\{i \in \Lambda(x)\colon\ ct_i(x) \leqslant y\bigr\}.$$

The *minimal covering transformation* of x with respect to y, denoted by $mct_y^P(x)$, is the covering transformation of x with respect to the minimal label:

$$mct_y^P(x) = ct_{mi}^P(x).$$

It is useful to state the following, which is a consequence of the definitions.

Proposition 2.4. *Let $x, y \in P$, with $x < y$. Then*

$$x \lhd mct_y^P(x) \leqslant y.$$

By Proposition 2.4, the following definition is well-posed.

Definition 2.5. Let $x, y \in P$, with $x < y$. The *minimal chain* from x to y is the saturated chain

$$x = x_0 \lhd x_1 \lhd \cdots \lhd x_k = y,$$

defined by

$$x_i = mct_y^P(x_{i-1}),$$

for every $i \in [k]$.

By the definition of minimal covering transformation, this chain has, among all the saturated chains from x to y, the lexicographically minimal labeling. The minimal chains are crucial in the definition of the *EL*-shellability: a poset is *EL*-shellable if its minimal chains have increasing labels and if any other saturated chain in it has at least one decrease in the labels.

2.2. *Coxeter groups and Bruhat order*

We refer to [12] for the definition of a Coxeter group. Let W be a Coxeter group, with set of generators S. The *length* of an element $w \in W$, denoted by $l(w)$, is the minimal k such that w can be written as a product of k generators. A *reflection* in a Coxeter group W is a conjugate of some element in S. The set of all reflections is usually denoted by T:

$$T = \{wsw^{-1}\colon\ s \in S,\ w \in W\}.$$

The *absolute length* of an element $w \in W$, denoted by $al(w)$, is the minimal k such that w can be written as the product of k reflections.

Let $u, v \in W$. We set $u \to v$ if and only if $v = ut$, with $t \in T$, and $l(u) < l(v)$. The *Bruhat order* of W is the partial order relation so defined: given $u, v \in W$, then $u \leqslant v$ if and only if there is a chain

$$u = u_0 \to u_1 \to u_2 \to \cdots \to u_k = v.$$

It is known that the map which associates with every element $w \in W$ its inverse w^{-1} is an automorphism of the Bruhat order, as we state in the following.

Proposition 2.6. *Let W be a Coxeter group and let $u, v \in W$. Then the following are equivalent*:

(1) $u \leqslant v$;
(2) $u^{-1} \leqslant v^{-1}$.

If W is finite it is known that W has a maximum, which is usually denoted by w_0. This element is an involution: $w_0^2 = 1$. Moreover, composition and conjugacy with w_0 induce (anti)automorphisms of the Bruhat order, as we state in the following.

Proposition 2.7. *Let W be a finite Coxeter group, with maximum w_0, and let $u, v \in W$. Then the following are equivalent*:

(1) $u \leqslant v$;
(2) $w_0 v \leqslant w_0 u$;
(3) $v w_0 \leqslant u w_0$;
(4) $w_0 u w_0 \leqslant w_0 v w_0$.

Bruhat order on Coxeter groups has been studied extensively. (see, e.g., [5–8,17,18,25]). In particular it is known that every Coxeter group, partially ordered by the Bruhat order, is a graded, *EL*-shellable poset (see [5,7,8,17]), which is also Eulerian (see [25]).

2.3. Classical Weyl groups

The finite irreducible Coxeter groups have been completely classified (see, e.g., [2,12]). Among them we find the classical Weyl groups, which have nice combinatorial descriptions in terms of permutation groups: the symmetric group S_n is a representative for type A, the hyperoctahedral group B_n for type B and the even-signed permutation group D_n for type D.

2.3.1. The symmetric group

We denote by S_n the *symmetric group*, defined by

$$S_n = \{\sigma : [n] \to [n]\colon\ \sigma \text{ is a bijection}\}$$

and we call its elements *permutations*. To denote a permutation $\sigma \in S_n$ we often use the *one-line notation*: we write $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, to mean that $\sigma(i) = \sigma_i$ for every $i \in [n]$. We also write σ in *disjoint cycle form*, omitting to write the 1-cycles: for example, if $\sigma = 364152$, then we also write $\sigma = (1,3,4)(2,6)$. Given $\sigma, \tau \in S_n$, we let $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1,2)(2,3) = (1,2,3)$. Given $\sigma \in S_n$, the *diagram* of σ is a square of $n \times n$ cells, with the cell (i, j) (that is, the cell in column i and row j, with the convention that the first column is the leftmost one and the first row is the lowest one) filled with a dot if and only if $\sigma(i) = j$. For example, in Fig. 1 the diagram of $\sigma = 35124 \in S_5$ is represented.

The *diagonal* of the diagram is the set of cells $\{(i,i)\colon\ i \in [n]\}$.

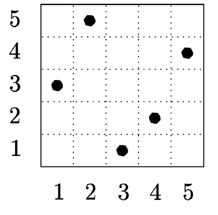


Fig. 1. Diagram of $\sigma = 35124 \in S_5$.

As a set of generators for S_n, we take $S = \{s_1, s_2, \ldots, s_{n-1}\}$, where $s_i = (i, i+1)$ for every $i \in [n-1]$. It is known that the symmetric group S_n, with this set of generators, is a Coxeter group of type A (see, e.g., [2, Proposition 1.5.4]).

The length of a permutation $\sigma \in S_n$ is given by $l(\sigma) = inv(\sigma)$, where

$$inv(\sigma) = \left|\left\{(i, j) \in [n]^2\colon\ i < j,\ \sigma(i) > \sigma(j)\right\}\right|$$

is the number of *inversions* of σ.

In the symmetric group the reflections are the transpositions:

$$T = \left\{(i, j) \in [n]^2 : i < j\right\}.$$

In order to give a characterization of the covering relation in the Bruhat order of the symmetric group, we give the following definition.

Let $\sigma \in S_n$. A *rise* of σ is a pair $(i, j) \in [n]^2$ such that $i < j$ and $\sigma(i) < \sigma(j)$. A rise (i, j) is said to be *free* if there is no $k \in [n]$ such that $i < k < j$ and $\sigma(i) < \sigma(k) < \sigma(j)$.

For example, the rises of $\sigma = 35124 \in S_5$ are $(1, 2)$, $(1, 5)$, $(3, 4)$, $(3, 5)$ and $(4, 5)$. They are all free except $(3, 5)$. The following is a well-known result.

Proposition 2.8. *Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Then $\sigma \lhd \tau$ in S_n if and only if*

$$\tau = \sigma(i, j),$$

where (i, j) is a free rise of σ.

The *standard labeling* of S_n is defined by associating with every pair $(\sigma, \tau) \in S_n^2$, with $\sigma \lhd \tau$, the pair (i, j) mentioned in Proposition 2.8, which is obviously unique.

In order to give a characterization of the Bruhat order relation in S_n, we introduce the following notation: for $\sigma \in S_n$ and for $(h, k) \in [n]^2$, we set

$$\sigma[h, k] = \left|\left\{i \in [h]\colon\ \sigma(i) \in [k, n]\right\}\right|.$$

The characterization is the following (see, e.g., [2, Theorem 2.1.5]).

Theorem 2.9. *Let $\sigma, \tau \in S_n$. Then $\sigma \leqslant \tau$ if and only if*

$$\sigma[h, k] \leqslant \tau[h, k],$$

for every $(h, k) \in [n]^2$.

Finally, the maximum of S_n is

$$w_0 = n(n-1)(n-2)\ldots 321.$$

Note that, given $\sigma \in S_n$, the diagrams of the permutations σ^{-1}, $w_0\sigma$, σw_0 and $w_0\sigma w_0$ are obtained from the diagram of σ by, respectively, interchanging rows and columns (transposing), reversing the rows, reversing the columns and reversing both rows and columns. So the effects of these operations on the Bruhat order are described in Propositions 2.6 and 2.7.

2.3.2. The hyperoctahedral group

We denote by $S_{\pm n}$ the symmetric group on the set $[\pm n]$:

$$S_{\pm n} = \{\sigma : [\pm n] \to [\pm n]\colon \ \sigma \text{ is a bijection}\}$$

(clearly isomorphic to S_{2n}), and by B_n the *hyperoctahedral group*, defined by

$$B_n = \{\sigma \in S_{\pm n}\colon \ \sigma(-i) = -\sigma(i) \text{ for every } i \in [n]\}$$

and we call its elements *signed permutations*. To denote a signed permutation $\sigma \in B_n$ we use the *window notation*: we write $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$, to mean that $\sigma(i) = \sigma_i$ for every $i \in [n]$ (the images of the negative entries are then uniquely determined). We also denote σ by the sequence $|\sigma_1||\sigma_2|\ldots|\sigma_n|$, with the negative entries underlined. For example, $\underline{3}\,\underline{2}\,1$ denotes the signed permutation $[-3, -2, 1]$. We also write σ in disjoint cycle form. Signed permutations are particular permutations of the set $[\pm n]$, so they inherit the notion of diagram. Note that the diagram of a signed permutation is symmetric with respect to the center. In Fig. 2, the diagram of $\sigma = \underline{3}\,\underline{2}\,1 \in B_3$ is represented.

The (*main*) *diagonal* of the diagram is the set of cells $\{(i,i)\colon \ i \in [\pm n]\}$, and the *antidiagonal* is the set of cells $\{(i,-i)\colon \ i \in [\pm n]\}$.

As a set of generators for B_n, we take $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 = (1,-1)$ and $s_i = (i, i+1)(-i, -i-1)$ for every $i \in [n-1]$. It is known that the hyperoctahedral group B_n, with this set of generators, is a Coxeter group of type B (see, e.g. [2, Proposition 8.1.3]).

The length of a signed permutation $\sigma \in B_n$ is given by

$$l(\sigma) = \frac{inv(\sigma) + neg(\sigma)}{2},$$

(see [14]), where

$$inv(\sigma) = \left|\{(i,j) \in [\pm n]^2\colon \ i < j, \ \sigma(i) > \sigma(j)\}\right|$$

(the length of σ in the symmetric group $S_{\pm n}$), and

$$neg(\sigma) = \left|\{i \in [n]\colon \ \sigma(i) < 0\}\right|.$$

For example, for $\sigma = \underline{3}\,\underline{2}\,1 \in B_3$, we have $inv(\sigma) = 8$, $neg(\sigma) = 2$, so $l(\sigma) = 5$.

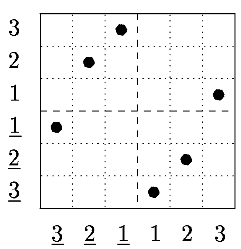


Fig. 2. Diagram of $\sigma = \underline{3}\,\underline{2}\,1 \in B_3$.

It is known (see, e.g., [2]) that the set of reflections of B_n is

$$T = \{(i, -i):\ i \in [n]\} \cup \{(i, j)(-i, -j):\ 1 \leqslant i < |j| \leqslant n\}.$$

In [15] we have found a characterization of the covering relation in B_n, for which we need the following definition.

Definition 2.10. Let $\sigma \in B_n$. A rise (i, j) of σ is *central* if

$$(0, 0) \in [i, j] \times [\sigma(i), \sigma(j)].$$

A central rise (i, j) of σ is *symmetric* if $j = -i$.

The characterization is the following.

Theorem 2.11. *Let $\sigma, \tau \in B_n$. Then $\sigma \lhd \tau$ if and only if either*

(1) *$\tau = \sigma(i, j)(-i, -j)$, where (i, j) is a non-central free rise of σ, or*
(2) *$\tau = \sigma(i, j)$, where (i, j) is a central symmetric free rise of σ.*

Theorem 2.11 is illustrated in Fig. 3, where black and white circles denote respectively σ and τ, inside the gray areas there are no other dots of σ and τ, and the diagrams of the two permutations are supposed to be the same anywhere else.

The *standard labeling* of B_n is defined by associating with every pair $(\sigma, \tau) \in B_n^2$, with $\sigma \lhd \tau$, the pair (i, j) mentioned in Theorem 2.11, which is obviously unique.

It is useful to extend a notation introduced for the symmetric group: for $\sigma \in B_n$ and for $(h, k) \in [\pm n]^2$ we set

$$\sigma[h, k] = \left|\{i \in [-n, h]:\ \sigma(i) \in [k, n]\}\right|.$$

Definition 2.12. Let $\sigma, \tau \in B_n$. We say that the pair (σ, τ) satisfies the *B-condition* if

$$\sigma[h, k] \leqslant \tau[h, k]$$

for every $(h, k) \in [\pm n]^2$.

The following result gives a combinatorial characterization of the Bruhat order relation in B_n (see, e.g. [2, Theorem 8.1.8]).

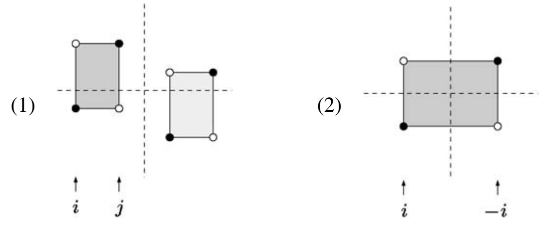


Fig. 3. Covering relation in B_n.

Theorem 2.13. *Let $\sigma, \tau \in B_n$. Then $\sigma \leqslant \tau$ if and only if the pair (σ, τ) satisfies the B-condition.*

Comparing Theorems 2.9 and 2.13, we can conclude that $\sigma \leqslant \tau$ in the Bruhat order of B_n if and only if $\sigma \leqslant \tau$ in the Bruhat order of the symmetric group $S_{\pm n}$. The maximum of B_n is

$$w_0 = \underline{1}\,\underline{2} \ldots \underline{n}$$

and the effects on the diagram of a signed permutation of taking the inverse, composing and conjugating with w_0 are the same as described for the symmetric group. Note that, by symmetry, some of these effects are trivial.

2.3.3. The even-signed permutation group

We denote by D_n the *even-signed permutation group*, defined by

$$D_n = \{\sigma \in B_n\colon \mathit{neg}(\sigma) \text{ is even}\}.$$

Notation and terminology are inherited from the hyperoctahedral group. For example the signed permutation $\sigma = \underline{3}\,\underline{2}\,1$, whose diagram is represented in Fig. 2, is also in D_3.

As a set of generators for D_n, we take $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 = (1, -2)(-1, 2)$ and $s_i = (i, i+1)(-i, -i-1)$ for every $i \in [n-1]$. It is known that the even-signed permutation group D_n, with this set of generators, is a Coxeter group of type D (see, e.g., [2, Proposition 8.2.3]).

The length of $\sigma \in D_n$ is given by

$$l(\sigma) = \frac{\mathit{inv}(\sigma) - \mathit{neg}(\sigma)}{2}$$

(see [15]), where $\mathit{inv}(\sigma)$ and $\mathit{neg}(\sigma)$ are defined as in the hyperoctahedral group. For example, for $\sigma = \underline{3}\,\underline{2}\,1 \in D_3$, we have $l(\sigma) = 3$.

Finally, it is known (see, e.g., [2]) that the set of reflections of D_n is

$$T = \big\{(i, j)(-i, -j)\colon 1 \leqslant i < |j| \leqslant n\big\}.$$

In [15] we have found a characterization of the covering relation in D_n, for which we need the following definition.

Definition 2.14. Let $\sigma \in D_n$. A central rise (i, j) is *semi-free* if

$$\big\{k \in [i, j]\colon \sigma(k) \in \big[\sigma(i), \sigma(j)\big]\big\} = \{i, -j, j\},$$

that is if the only dots of the diagram of σ lying in the rectangle $[i, j] \times [\sigma(i), \sigma(j)]$ are exactly those in the cells $(i, \sigma(i))$, $(-j, -\sigma(j))$ and $(j, \sigma(j))$.

An example of a central semi-free rise is illustrated in Fig. 4(3).

Theorem 2.15. *Let $\sigma, \tau \in D_n$. Then $\sigma \lhd \tau$ if and only if*

$$\tau = \sigma(i, j)(-i, -j),$$

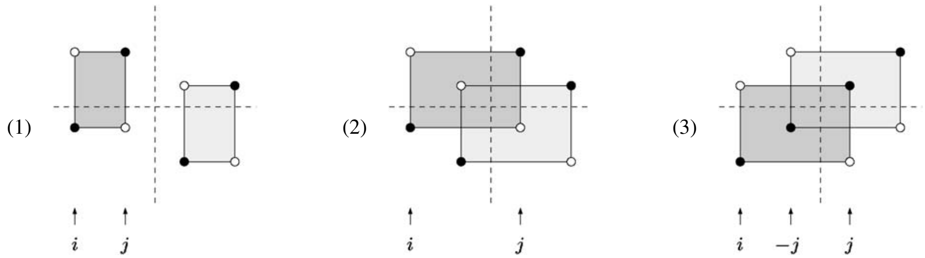


Fig. 4. Covering relation in D_n.

where (i, j) *is*

(1) *a non-central free rise of* σ*, or*
(2) *a central non-symmetric free rise of* σ*, or*
(3) *a central semi-free rise of* σ.

Theorem 2.15 is illustrated in Fig. 4, where we use the same notation as in Fig. 3.

The *standard labeling* of D_n is defined by associating with every pair $(\sigma, \tau) \in D_n^2$, with $\sigma \lhd \tau$, the pair (i, j) mentioned in Theorem 2.15, which is obviously unique.

In order to give a combinatorial characterization of the Bruhat order relation in D_n, we introduce the following notation: for $\sigma \in D_n$ and $A, B \subseteq [\pm n]$, we set

$$\sigma_{A\times B} = \left|\left\{i \in A\colon\ \sigma(i) \in B\right\}\right|.$$

Then, for $\sigma \in D_n$ and $(h, k) \in [-n] \times [n]$, we set

$$\begin{aligned}
\sigma_{center}[h, k] &= \sigma_{[\pm|h|]\times[\pm k]},\\
\sigma_{NW}[h, k] &= \sigma_{[-n,h-1]\times[k+1,n]},\\
\sigma_{Nleft}[h, k] &= \sigma_{[h]\times[k+1,n]},\\
\sigma_{Wup}[h, k] &= \sigma_{[-n,h-1]\times[k]}.
\end{aligned}$$

We say that $(h, k) \in [-n] \times [n]$ is *free* for σ if

$$\sigma_{center}[h, k] = 0.$$

Definition 2.16. Let $\sigma, \tau \in D_n$. We say that $(h, k) \in [-n] \times [n]$ is a *D-cell* of the pair (σ, τ) if it is free for both σ and τ and

$$\sigma_{NW}[h, k] = \tau_{NW}[h, k].$$

If (h, k) is a D-cell of (σ, τ), then we say that it is *valid* if

$$\sigma_{Nleft}[h, k] \equiv \tau_{Nleft}[h, k],$$

or, equivalently, if

$$\sigma_{Wup}[h,k] \equiv \tau_{Wup}[h,k].$$

Finally, we say that the pair (σ,τ) satisfies the *D-condition* if every D-cell of (σ,τ) is valid.

The following result gives a combinatorial characterization of the Bruhat order relation in D_n (see [2, Theorem 8.2.8]).

Theorem 2.17. *Let $\sigma,\tau \in D_n$. Then $\sigma \leqslant \tau$ if and only if the pair (σ,τ) satisfies both the B-condition and the D-condition.*

Note that $\sigma \leqslant \tau$ in D_n implies $\sigma \leqslant \tau$ in B_n, while the converse is not true.
The maximum of D_n is

$$w_0 = \begin{cases} \underline{1}\,\underline{2}\ldots\underline{n}, & \text{if } n \text{ is even,}\\ 1\underline{2}\ldots\underline{n}, & \text{if } n \text{ is odd.}\end{cases}$$

We now briefly recall the main results obtained in [15] about the minimal covering transformation in classical Weyl groups, as they are needed in the rest of this work. For these posets the set of labels is $\{(i,j) \in I^2\colon\ i<j\}$, where $I=[n]$ for S_n and $I=[\pm n]$ for B_n and D_n, totally ordered by the lexicographic order.

Definition 2.18. Let $\sigma,\tau \in S_n$, with $\sigma<\tau$. The *difference index* of σ with respect to τ, denoted by $di_\tau(\sigma)$ (or simply di), is the minimal index on which σ and τ differ:

$$di_\tau(\sigma) = \min\bigl\{i \in [n]\colon\ \sigma(i) \neq \tau(i)\bigr\}.$$

The *covering index* of σ with respect to τ, denoted by $ci_\tau(\sigma)$ (or simply ci), is

$$ci_\tau(\sigma) = \min\bigl\{j \in [di+1,n]\colon\ \sigma(j) \in \bigl[\sigma(di)+1, \tau(di)\bigr]\bigr\}.$$

The following has been proved in [15, Section 4].

Theorem 2.19. *Let $\sigma,\tau \in S_n$, with $\sigma<\tau$. The* minimal label *of σ with respect to τ is*

$$mi_\tau(\sigma) = (di, ci).$$

The minimal covering transformation *of σ with respect to τ is*

$$mct_\tau^{S_n}(\sigma) = \sigma(di, ci).$$

For the hyperoctahedral group the situation is the following ([15, Section 5]). Note that the notions of difference index and covering index are inherited by the groups of signed permutations.

Theorem 2.20. *Let $\sigma, \tau \in B_n$, with $\sigma < \tau$. The* minimal label *of σ with respect to τ is*

$$mi_\tau(\sigma) = \begin{cases} (di, ci), & \text{if } (di, ci) \text{ is non-central}, \\ (di, -di), & \text{if } (di, ci) \text{ is central}. \end{cases}$$

The minimal covering transformation *of σ with respect to τ is*

$$mct_\tau^{B_n}(\sigma) = \begin{cases} \sigma(di, ci)(-di, -ci), & \text{if } (di, ci) \text{ is non-central}, \\ \sigma(di, -di), & \text{if } (di, ci) \text{ is central}. \end{cases}$$

For the even-signed permutation group, we introduce the following definition, which can be given in general for the symmetric group, and which the hyperoctahedral group and the even-signed permutation group inherit.

Definition 2.21. Let $\sigma, \tau \in S_n$, with $\sigma < \tau$. Suppose that the set

$$\bigl\{ j \in [ci+1, n]\colon\ \sigma(j) \in \bigl[\sigma(di)+1, \sigma(ci)-1\bigr]\bigr\}$$

is not empty. Then the *second covering index* of σ with respect to τ, denoted by $sci_\tau(\sigma)$ (or simply *sci*) is

$$sci_\tau(\sigma) = \min\bigl\{ j \in [ci+1, n]\colon\ \sigma(j) \in \bigl[\sigma(di)+1, \sigma(ci)-1\bigr]\bigr\}.$$

Definition 2.22. Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. We say that (σ, τ) is a *D-special pair* if

(1) $(di <)\ ci < 0$;
(2) $(\sigma(di) <)\ \sigma(ci) < 0$;
(3) $\tau(di) = -\sigma(ci)$;
(4) $[ci+1, -ci-1] \times [\sigma(ci), -\sigma(ci)]$ is empty for σ.

Moreover, a D-special pair (σ, τ) can be either *of the first kind*, if

(5′) $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ is not empty for σ,

or *of the second kind*, if

(5″) $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ is empty for σ.

The following has been proved in [15, Section 6].

Theorem 2.23. *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. The* minimal label *of σ with respect to τ is*

$$mi_\tau(\sigma) = \begin{cases} (di, ci), & \text{if } (\sigma, \tau) \text{ is not a } D\text{-special pair}, \\ (di, sci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the first kind}, \\ (di, -ci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the second kind}, \end{cases}$$

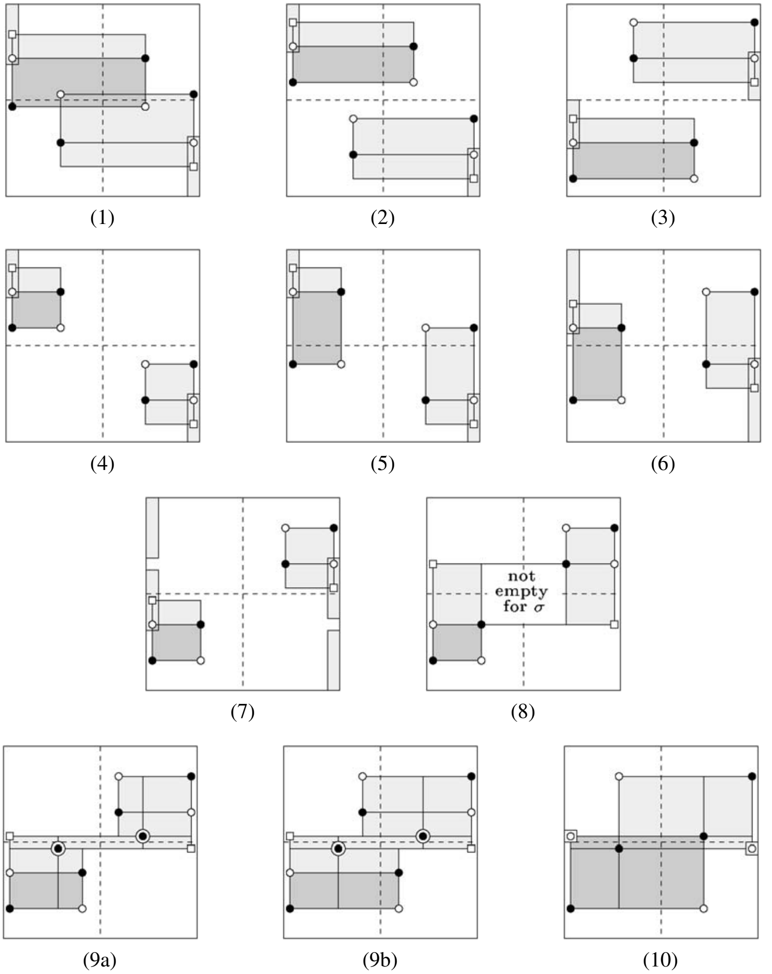


Fig. 5. Minimal covering transformation in D_n.

The minimal covering transformation *of* σ *with respect to* τ *is*

$$mct_\tau^{D_n}(\sigma) = \begin{cases} \sigma(di, ci)(-di, -ci), & \text{if } (\sigma, \tau) \text{ is not a } D\text{-special pair},\\ \sigma(di, sci)(-di, -sci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the first kind},\\ \sigma(di, -ci)(-di, ci), & \text{if } (\sigma, \tau) \text{ is a } D\text{-special pair of the second kind}.\end{cases}$$

All cases are shown in Fig. 5, where σ, τ and $\chi = mct_\tau^{D_n}(\sigma)$ are represented. Black circles denote σ, white squares τ and white circles χ. Only the dots in columns di and $-di$ of τ are represented, possibly with a gray rectangle around, denoting the range of variation of $\tau(di)$. Inside the gray rectangles there are no other dots of σ and χ than those indicated and the diagrams of σ and χ are supposed to be the same anywhere else.

If (σ, τ) is not a D-special pair, we distinguish between the following cases:

(1) $(di <)\ 0 < ci$, $\sigma(di) < 0 < \sigma(ci)$;
(2) $(di <)\ 0 < ci$, $0 < \sigma(di)\ (< \sigma(ci))$;

(3) $(di <)\ 0 < ci$, $(\sigma(di) <)\ \sigma(ci) < 0$;
(4) $(di <)\ ci < 0$, $0 < \sigma(di)\ (< \sigma(ci))$;
(5) $(di <)\ ci < 0$, $\sigma(di) < 0 < \sigma(ci)$, $\sigma(ci) > -\sigma(di)$;
(6) $(di <)\ ci < 0$, $\sigma(di) < 0 < \sigma(ci)$, $\sigma(ci) < -\sigma(di)$;
(7) $(di <)\ ci < 0$, $(\sigma(di) <)\ \sigma(ci) < 0$, $\tau(di) \neq -\sigma(ci)$;
(8) $(di <)\ ci < 0$, $(\sigma(di) <)\ \sigma(ci) < 0$, $\tau(di) = -\sigma(ci)$, $[ci+1, -ci-1] \times [\sigma(ci), -\sigma(ci)]$ is not empty for σ.

Otherwise (σ, τ) can be either a D-special pair of the first kind:

(9) $(di <)\ ci < 0$, $(\sigma(di) <)\ \sigma(ci) < 0$, $\tau(di) = -\sigma(ci)$, $[ci+1, -ci-1] \times [\sigma(ci), -\sigma(ci)]$ is empty for σ, but $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ is not, and we distinguish between
  (9a) $sci < 0$ and
  (9b) $sci > 0$;

or a D-special pair of the second kind:

(10) $(di <)\ ci < 0$, $(\sigma(di) <)\ \sigma(ci) < 0$, $\tau(di) = -\sigma(ci)$, $[ci+1, -ci-1] \times [\sigma(di), -\sigma(di)]$ is empty for σ.

3. Absolute length of involutions in classical Weyl groups

In classical Weyl groups there is a nice combinatorial description for the absolute length of the involutions, as we show in this section.

Let $\sigma \in S_n$. The number of *exceedances* of σ is

$$exc(\sigma) = \left|\left\{i \in [n]:\ \sigma(i) > i\right\}\right|,$$

that is, the number of dots of the diagram which are above the diagonal.

In the symmetric group the absolute length of an involution is simply given by the number of exceedances.

Proposition 3.1. *Let $\sigma \in Invol(S_n)$. Then*

$$al(\sigma) = exc(\sigma).$$

Proof. Let $\{i_1, \dots, i_e\}$ be the exceedances of σ. If we set $j_p = \sigma(i_p)$, for $p \in [e]$, then we have

$$\sigma = (i_1, j_1) \dots (i_e, j_e). \tag{1}$$

Since $al(\sigma)$ is the minimal number of transpositions in which σ can be decomposed, it follows that

$$al(\sigma) \leqslant exc(\sigma).$$

On the other hand, for every $\chi \in S_n$ and for every transposition (i, j) we have $exc(\chi(i, j)) \leqslant exc(\chi) + 1$, as it can be easily checked. So

$$exc(\sigma) \leqslant al(\sigma).$$

Thus $al(\sigma) = exc(\sigma)$ and (1) gives a minimal decomposition of σ as a product of reflections. □

For example, for $\sigma = 32154 \in Invol(S_5)$, we have $al(\sigma) = exc(\sigma) = 2$. In fact

$$\sigma = \underbrace{(1, 3)}_{t_1} \cdot \underbrace{(4, 5)}_{t_2}$$

is a minimal decomposition of σ as a product of reflections of S_5.

In [14] we have introduced the statistic dna on a signed permutation.

Definition 3.2. Let $\sigma \in B_n$. The number of *deficiencies-not-antideficiencies* of σ is

$$dna(\sigma) = \left|\left\{i \in [n]\colon\ -i \leqslant \sigma(i) < i\right\}\right|,$$

that is, the number of dots of the diagram of σ which are below the main diagonal (deficiencies) and not below the antidiagonal (not-antideficiencies).

For example, consider the signed permutation $\sigma = 4\,\underline{7}\,\underline{3}\,1\,5\,\underline{6}\,\underline{2} \in B_7$, whose diagram is shown in Fig. 6. Looking at the picture, $dna(\sigma)$ is the number of dots which lie in the gray area, that is $dna(\sigma) = 4$.

A surprising fact is that in the hyperoctahedral group and in the even-signed permutation group, the combinatorial description for the absolute length of an involution is exactly the same: in both cases it is given by the dna statistic. But for different reasons.

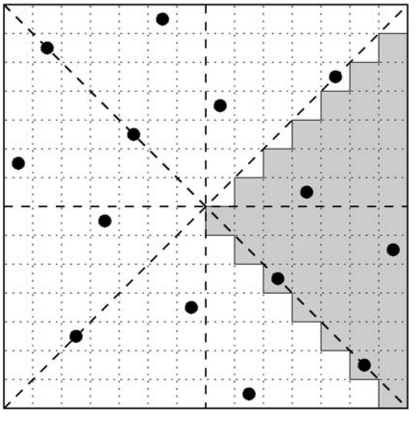


Fig. 6. The dna statistic.

Proposition 3.3. *Let $\sigma \in Invol(B_n)$. Then*

$$al(\sigma) = dna(\sigma).$$

Proof. Let $\{i_1, \dots, i_r\}$ be the deficiencies-antiexceedances of σ, that is, the indices $i > 0$ such that $-i < \sigma(i) < i$. Let $\{h_1, \dots, h_s\}$ be the positive antifixed points of σ, that is, the indices $i > 0$ such that $\sigma(i) = -i$. Obviously $dna(\sigma) = r + s$. If we set $j_p = \sigma(i_p)$, for every $p \in [r]$, then we have

$$\sigma = \prod_{p=1}^{r} (i_p, j_p)(-i_p, -j_p) \cdot \prod_{q=1}^{s} (h_q, -h_q). \tag{2}$$

Since $t_p = (i_p, j_p)(-i_p, -j_p)$, for $p \in [r]$, and $t_{r+q} = (h_q, -h_q)$, for $q \in [s]$, are all reflections of B_n, we have

$$al(\sigma) \leqslant r + s = dna(\sigma).$$

On the other hand, for every $\chi \in B_n$ and for every reflection t of B_n we have $dna(\chi t) \leqslant dna(\chi) + 1$. So

$$dna(\sigma) \leqslant al(\sigma).$$

Thus $al(\sigma) = dna(\sigma)$ and (2) gives a minimal decomposition of σ as a product of reflections. □

For example, for the involution of Fig. 6, we have $al(\sigma) = dna(\sigma) = 4$. In fact

$$\sigma = \underbrace{(1,4)(-1,-4)}_{t_1} \cdot \underbrace{(7,-2)(-7,2)}_{t_2} \cdot \underbrace{(3,-3)}_{t_3} \cdot \underbrace{(6,-6)}_{t_4} \tag{3}$$

is a minimal decomposition of σ as a product of reflections of B_7.

Proposition 3.4. *Let $\sigma \in Invol(D_n)$. Then*

$$al(\sigma) = dna(\sigma).$$

Proof. Let $\{i_1, \dots, i_r\}$ be the deficiencies-antiexceedances of σ. Note that an even-signed permutation which is an involution must have an even number of positive antifixed points, so we can consider them in pairs: let $\{h_1, k_1, \dots, h_s, k_s\}$ be the positive antifixed points of σ. We now have $dna(\sigma) = r + 2s$. If we set $j_p = \sigma(i_p)$, for every $p \in [r]$, then we have

$$\sigma = \prod_{p=1}^{r} (i_p, j_p)(-i_p, -j_p) \cdot \prod_{q=1}^{s} (h_q, k_q)(-h_q, -k_q) \cdot \prod_{q=1}^{s} (h_q, -k_q)(-h_q, k_q). \tag{4}$$

Since $t_p = (i_p, j_p)(-i_p, -j_p)$, for $p \in [r]$, $t_{r+2q-1} = (h_q, k_q)(-h_q, -k_q)$ and $t_{r+2q} = (h_q, -k_q)(-h_q, k_q)$, for $q \in [s]$, are all reflections of D_n, we have

$$al(\sigma) \leqslant r + 2s = dna(\sigma).$$

On the other hand, for every $\chi \in B_n$ and for every reflection t of B_n we have $dna(\chi t) \leqslant dna(\chi) + 1$. So

$$dna(\sigma) \leqslant al(\sigma).$$

Thus $al(\sigma) = dna(\sigma)$ and (4) gives a minimal decomposition of σ as a product of reflections. □

For example, for the involution of Fig. 6, which is also in $Invol(D_7)$, we have $al(\sigma) = dna(\sigma) = 4$. Note that the decomposition in (3) does not work in D_7, since $(3, -3)$ and $(6, -6)$ are not elements of D_7. But

$$\sigma = \underbrace{(1,4)(-1,-4)}_{t_1} \cdot \underbrace{(7,-2)(-7,2)}_{t_2} \cdot \underbrace{(3,6)(-3,-6)}_{t_3} \cdot \underbrace{(3,-6)(-3,6)}_{t_4}$$

is a minimal decomposition of σ as a product of reflections of D_7.

4. The main result

The main results of this paper, together with the results in [13,14], can be summarized in the following.

Theorem 4.1. *Let W be a classical Weyl group. The poset $Invol(W)$ is*

(1) *graded, with rank function given by*

$$\rho(w) = \frac{l(w) + al(w)}{2},$$

for every $w \in Invol(W)$;
(2) *EL-shellable*;
(3) *Eulerian.*

Considering the description of the absolute length of involutions given in the preceding section, it turns out that Theorem 4.1 has already been proved for the symmetric group and for the hyperoctahedral group, respectively in [13,14].

In order to prove the result for the even-signed permutation group, it has been crucial to find a unified approach for the three families of classical Weyl groups (see [16]). This has led to a description of the results for S_n and B_n which is slightly different compared with how they are presented in [13,14].

In next section we describe the main concepts and results regarding the hyperoctahedral group, presented with this new approach, as they are needed in the last section, where we prove the result for the even-signed permutation group.

5. Bruhat order on the involutions of B_n

Looking at the diagram of a signed permutation, by *orbit* of an object (which can be a dot, a cell, or a rectangle of cells), we mean the set made of that object and its symmetric images with respect to the main diagonal, the antidiagonal and the center.

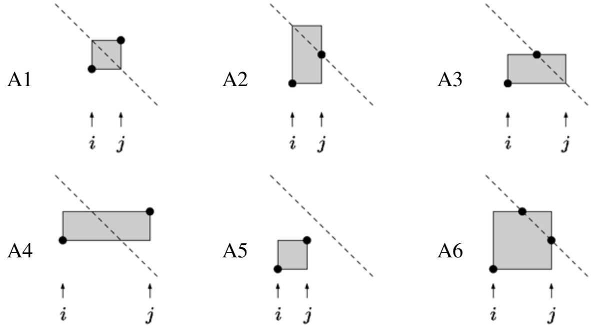


Fig. 7. Covering relation in *Invol*(B_n): anticases.

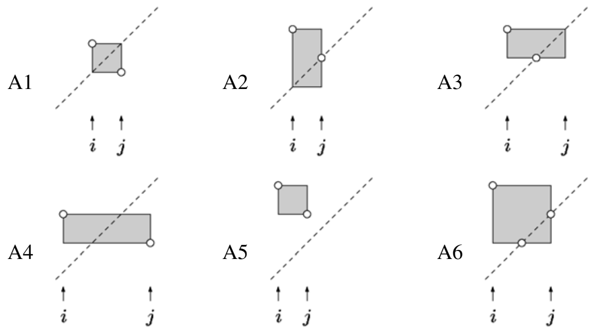


Fig. 8. Covering relation in *Invol*(B_n): main cases.

Definition 5.1. Let $\sigma, \tau \in Invol(B_n)$. We say that (σ, τ) is a *good pair* in *Invol*(B_n) if there exists a rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$ such that σ and τ have the same diagram except for the dots in R, and in the rectangles of its orbit, for which the situation, depending on the position of R with respect to the antidiagonal and to the main diagonal, is described in Figs. 7 and 8: black and white circles denote, respectively, σ and τ, and the rectangle R (gray rectangle) contains no other dots of σ and τ than those indicated.

The *case* of the pair (σ, τ) is (Ah, Mk), with $h, k \in [6]$, where Ah (*anticase*) and Mk (*main case*) refer to the pictures of Figs. 7 and 8.

Note that for geometrical reasons not all the 36 pairs are possible cases.

Let (σ, τ) be a good pair in *Invol*(B_n). The *main rectangle* of (σ, τ), denoted by $R(\sigma, \tau)$ is the rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$ mentioned in Definition 5.1.

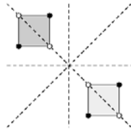
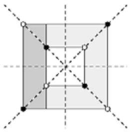
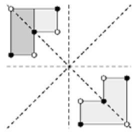
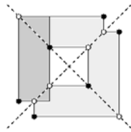
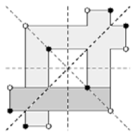
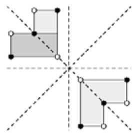
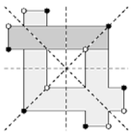
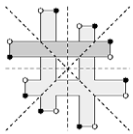
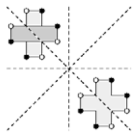
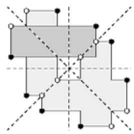
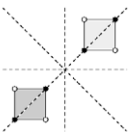
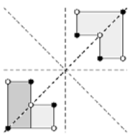
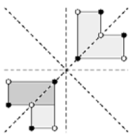
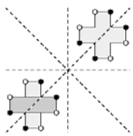
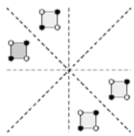
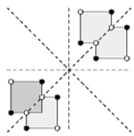
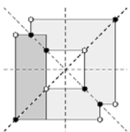
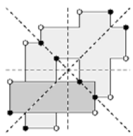
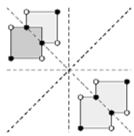
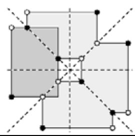
It is useful to consider separately the cases in which $R(\sigma, \tau)$ is *central*, that is $(0, 0) \in R(\sigma, \tau)$: non-central cases are illustrated in Table 1, and central cases in Table 2. In every case, black circles denote σ and white circles denote τ. The darker gray rectangle is the main rectangle $R(\sigma, \tau)$ and the complete gray area is the union of the rectangles of its orbit.

In Tables 1 and 2 almost all cases are illustrated. In fact, in some cases there is more than one possibility: see Table 3.

We set

$$H_{Invol(B_n)} = \{(\sigma, \tau) \in Invol(B_n)^2\colon\ (\sigma, \tau) \text{ is a good pair in } Invol(B_n)\},$$

Table 1
Covering relation in $Invol(B_n)$: non-central cases

	M1	M2	M3	M4	M5	M6
A1	–	–	–	–		–
A2	–		–	–		
A3	–	–	–			–
A4	–	–				
A5						
A6	–		–			

and define the *standard labeling* λ of $Invol(B_n)$ by associating, with every good pair $(\sigma, \tau) \in H_{Invol(B_n)}$, the pair $(i, j) \in [\pm n]^2$, if $R = [i, j] \times [\sigma(i), \tau(i)]$ is the main rectangle of (σ, τ).

By Theorem 2.11, $H_{Invol(B_n)}$ is a successor system and, since τ is uniquely determined by σ and by the label (i, j), λ is a good labeling.

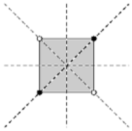
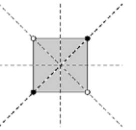
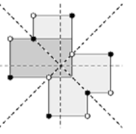
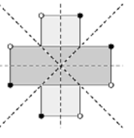
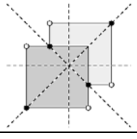
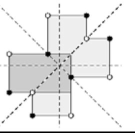
Given $\sigma \in Invol(B_n)$, a pair $(i, j) \in [\pm n]^2$ is a *suitable label* of σ if there exists $\tau \in Invol(B_n)$, with $\lambda(\sigma, \tau) = (i, j)$. Such a τ, obviously unique, is called the *transformation* of σ with respect to (i, j) and it is denoted by

$$t_{(i,j)}^{Invol(B_n)}(\sigma).$$

Definition 5.2. Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$. Consider the following seven properties of the pair (σ, τ):

(1) $ci \in \{-di, -\sigma(di)\}$;
(2) (di, ci) is a central rise of σ (not necessarily symmetric);

Table 2
Covering relation in $Invol(B_n)$: central cases

	M1	M2	M3	M4	M5	M6
A1		–	–	–	–	
A2	–	–	–	–	–	–
A3	–	–	–	–	–	
A4	–	–	–		–	–
A5	–	–	–	–	–	–
A6		–		–	–	–

(3) $\sigma(ci) \neq -ci$;
(4) $\tau(di) = -di$;
(5) *sci* does not exist, or $\sigma(sci) > -sci$;
(6) *sci* exists, and $sci = -ci$;
(7) *sci* exists, and $\sigma(sci) = -sci$.

The *B-type* of the pair (σ, τ) is

$$B\text{-}type(\sigma, \tau) = Bh,$$

where $h = 8$ if (σ, τ) does not satisfy any of the above properties, otherwise

$$h = \min\{k \in [7]:\ (\sigma, \tau) \text{ satisfies property } k\}.$$

All cases are represented in Tables 4 to 11. The notation used in the pictures will be soon described. But we first need the following definition.

Definition 5.3. Let $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$. The *B-covering index* of σ with respect to τ, denoted by $Bci_\tau(\sigma)$ (or simply Bci), is the minimal index $j \in [di + 1, n]$ such that there exists $\chi \in Invol(B_n)$, with

(1) $(\sigma, \chi) \in H_{Invol(B_n)}$;
(2) $\lambda(\sigma, \chi) = (di, j)$;
(3) $\chi(di) \leqslant \tau(di)$.

As it comes out from the pictures, Bci is always well defined and precisely:

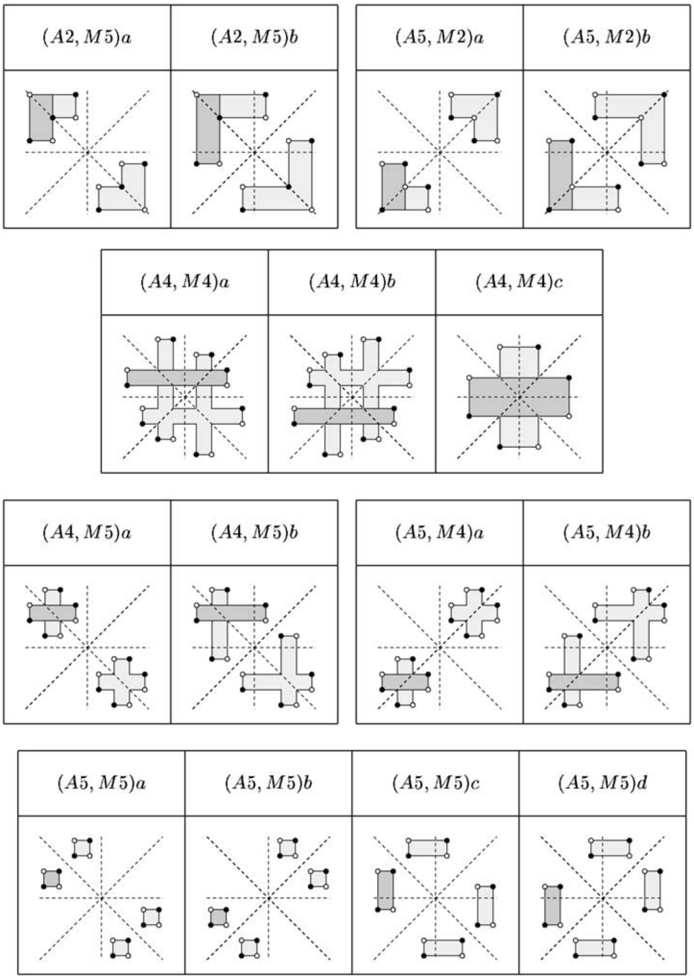
$$Bci = \begin{cases} ci, & \text{if } B\text{-}type(\sigma,\tau) = B1, B3, B4, \\ -di, & \text{if } B\text{-}type(\sigma,\tau) = B2, \\ -\sigma(di), & \text{if } B\text{-}type(\sigma,\tau) = B5, \\ sci, & \text{if } B\text{-}type(\sigma,\tau) = B6, B7, B8. \end{cases}$$

By definition, (di, Bci) is a suitable label of σ, so we can consider the involution

$$\chi_{Invol(B_n)}(\sigma,\tau) = t^{Invol(B_n)}_{(di,Bci)}(\sigma).$$

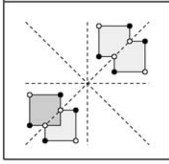
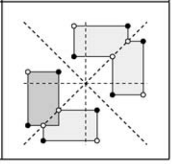
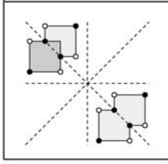
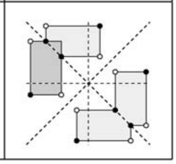
In the pictures the involutions σ, τ and $\chi = \chi_{Invol(B_n)}(\sigma,\tau)$ are represented: black circles denote σ, white squares denote τ and white circles denote χ. Only the dot in column di of τ,

Table 3
The cases of Tables 1 and 2 having more than one possibility



(*continued on next page*)

Table 3 (*continued*)

$(A5, M6)a$	$(A5, M6)b$	$(A6, M5)a$	$(A6, M5)b$
			

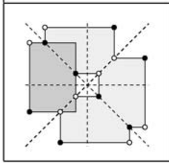
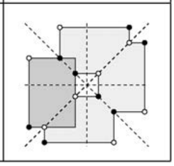
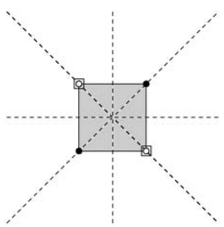
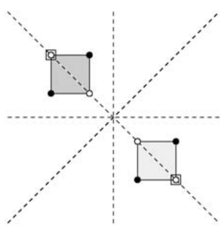
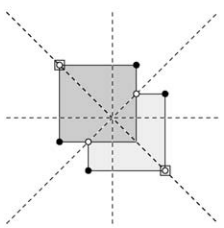
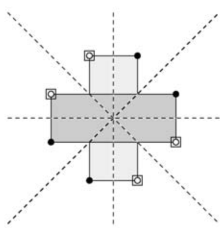
$(A6, M6)a$	$(A6, M6)b$
	

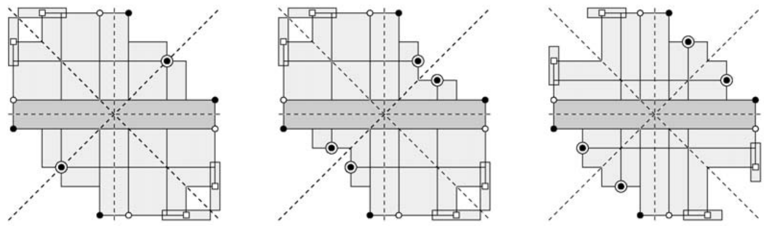
Table 4
B-type B1

B1	▸ $ci \in \{-di, -\sigma(di)\}$ $\qquad Bci = ci$
 B1.1	 B1.2
 B1.3	 B1.4

with its orbit, is indicated, possibly with a gray rectangle around, indicating the range of variation of $\tau(di)$. The involutions σ and χ are supposed to have the same diagram anywhere else. Finally the darker gray rectangle is the main rectangle of (σ, χ) and inside the complete gray area there are no other dots of σ (hence of χ) than those indicated.

Table 5
B-type B2

B2	▶ $ci \notin \{-di, -\sigma(di)\}$ ▶ $(0,0) \in [di, ci] \times [\sigma(di), \sigma(ci)]$	$Bci = -di$



B2.1 B2.2 B2.3

The following definitions are valid in general for signed permutations (not necessarily involutions).

Definition 5.4. Let $\sigma, \tau \in B_n$, with $\sigma < \tau$. We say that (σ, τ) is a *B-exceptional pair* if

(1) $\tau(di) < di$;
(2) $\tau(\tau(di)) = di$;
(3) $\sigma(ci) = -ci$;
(4) sci exists;
(5) $\sigma(sci) < -sci$;
(6) $[-\tau(di), sci] \times [\tau(di) + 1, -di]$ is empty for σ.

Examples of B-exceptional pairs are the pairs $(\sigma, \tau) \in Invol(B_n)^2$, with $\sigma < \tau$, whose B-type is $B8$ (see Table 11).

Definition 5.5. Let (σ, τ) be a B-exceptional pair. The *second covering transformation* in B_n of σ with respect to τ, denoted by $sct_\tau^{B_n}(\sigma)$ is

$$sct_\tau^{B_n}(\sigma) = \sigma(di, sci)(-di, -sci).$$

The following has been proved in [14].

Proposition 5.6. *Let (σ, τ) be a B-exceptional pair. Then*

$$sct_\tau^{B_n}(\sigma) \leqslant \tau.$$

In [16] it has been proved the insertion property of $Invol(B_n)$, namely, given $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$, then

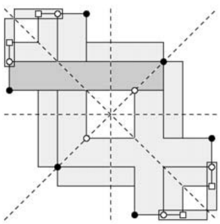
$$\chi_{Invol(B_n)}(\sigma, \tau) \leqslant \tau.$$

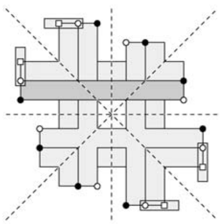
The average between the length and the absolute length of $\sigma \in Invol(B_n)$ is

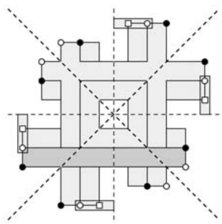
$$\rho(\sigma) = \frac{inv(\sigma) + neg(\sigma) + 2dna(\sigma)}{4}.$$

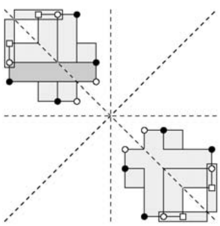
Table 6
B-type B3

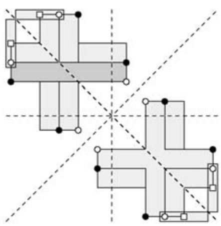
B3	▶ $ci \notin \{-di, -\sigma(di)\}$ ▶ $(0,0) \notin [di, ci] \times [\sigma(di), \sigma(ci)]$ ▶ $\sigma(ci) \neq -ci$	$Bci = ci$

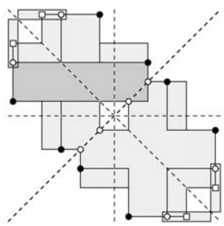

B3.1

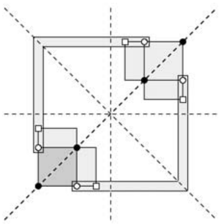

B3.2

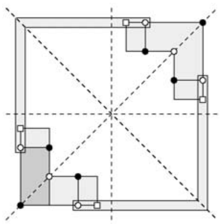

B3.3

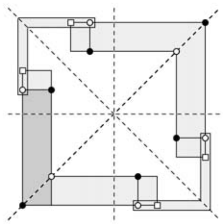

B3.4


B3.5


B3.6


B3.7

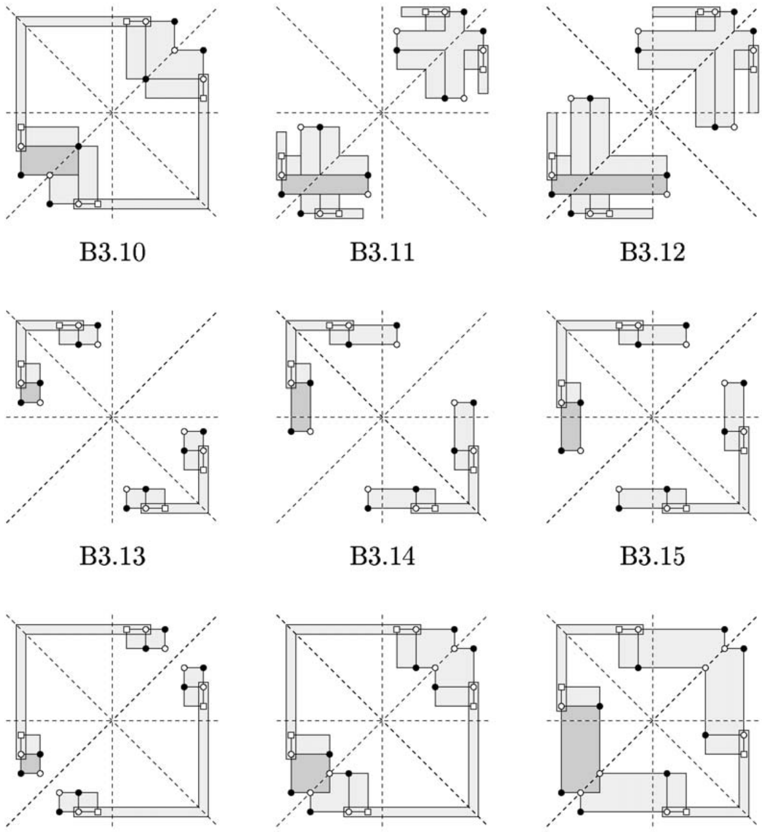

B3.8


B3.9

(*continued on next page*)

Table 6 (*continued*)

B3	▸ $ci \notin \{-di, -\sigma(di)\}$ ▸ $(0,0) \notin [di, ci] \times [\sigma(di), \sigma(ci)]$ ▸ $\sigma(ci) \neq -ci$	$Bci = ci$



B3.10 B3.11 B3.12

B3.13 B3.14 B3.15

B3.16 B3.17 B3.18

From the insertion property the following has been derived: the pair $(H_{Invol(B_n)}, \rho)$ is a covering system of $Invol(B_n)$. This implies the gradedness of $Invol(B_n)$, with rank function ρ, the same result obtained in [14].

We also have a characterization of the covering relation in $Invol(B_n)$: if $\sigma, \tau \in Invol(B_n)$, then $\sigma \lhd \tau$ if and only if (σ, τ) is a good pair in $Invol(B_n)$. And the transformation of σ with respect to a suitable label (i, j) actually is a *covering transformation*, denoted by

$$ct_{(i,j)}^{Invol(B_n)}(\sigma).$$

Table 7
B-type B4

B4	► $\sigma(ci) = -ci$ ► $\tau(di) = -di$	$Bci = ci$

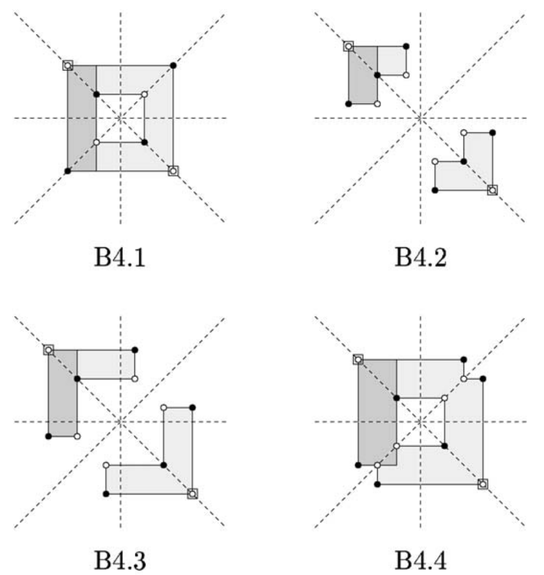


Table 8
B-type B5

B5	► $\sigma(ci) = -ci$ ► $\tau(di) < -di$ ► sci does not exist, or $\sigma(sci) > -sci$	$Bci = -\sigma(di)$

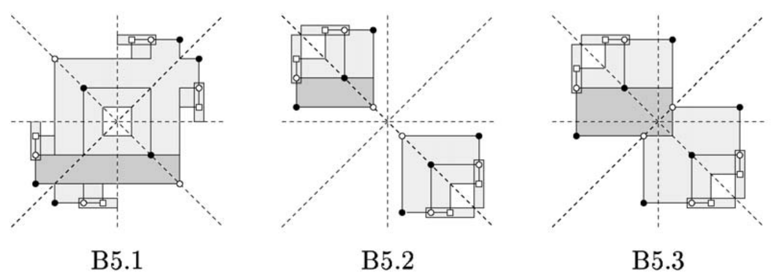
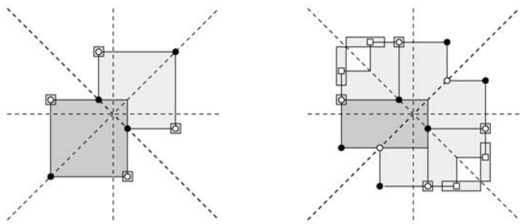


Table 9
B-type B6

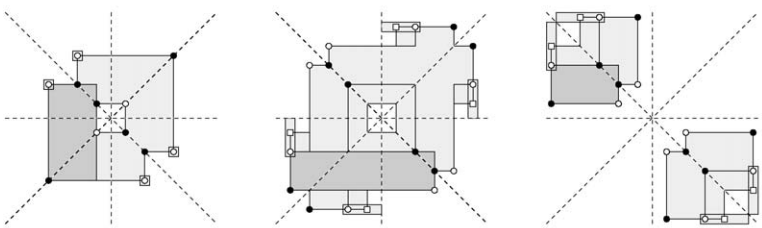
B6	▶ $\sigma(ci) = -ci$ ▶ $\tau(di) < -di$ ▶ sci exists ▶ $sci = -ci$	$Bci = sci$



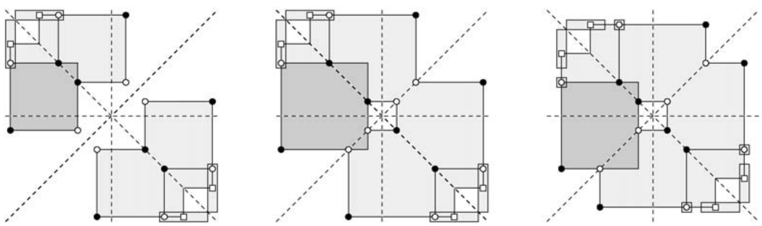
B6.1 B6.2

Table 10
B-type B7

B7	▶ $\sigma(ci) = -ci$ ▶ $\tau(di) < -di$ ▶ sci exists ▶ $\sigma(sci) = -sci$ ▶ $sci \neq -ci$	$Bci = sci$



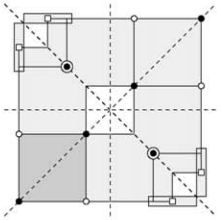
B7.1 B7.2 B7.3



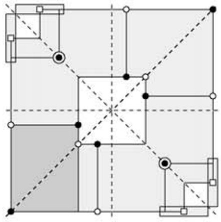
B7.4 B7.5 B7.6

Table 11
B-type B8

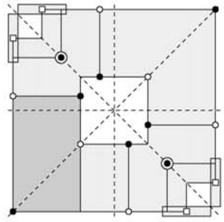
B8	▸ $\sigma(ci) = -ci$ ▸ $\tau(di) < -di$ ▸ sci exists ▸ $\sigma(sci) < -sci$	$Bci = sci$



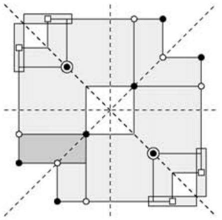
B8.1



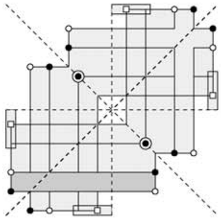
B8.2



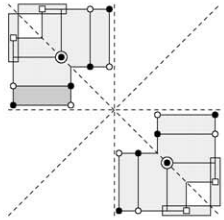
B8.3



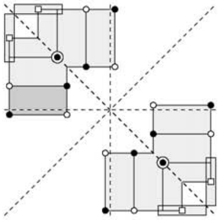
B8.4



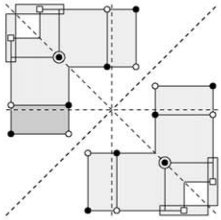
B8.5



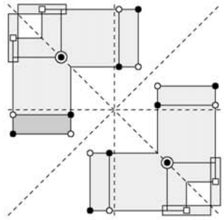
B8.6



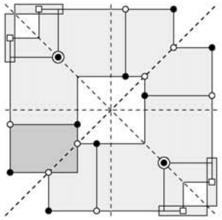
B8.7



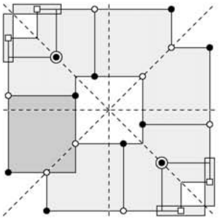
B8.8



B8.9



B8.10



B8.11

By the definitions of di and Bci, it follows that , given $\sigma, \tau \in Invol(B_n)$, with $\sigma < \tau$, the *minimal label* of σ with respect to τ is (di, Bci), so

$$mct_{\tau}^{Invol(B_n)}(\sigma) = ct_{(di,Bci)}^{Invol(B_n)}(\sigma) = \chi_{Invol(B_n)}(\sigma, \tau).$$

Starting from this it has been proved that $Invol(B_n)$ is *EL*-shellable and Eulerian (see [14,16]).

6. Bruhat order on the involutions of D_n

In this section we study the poset $Invol(D_n)$ of the involutions of D_n. In Fig. 9 the poset $Invol(D_4)$ is illustrated.

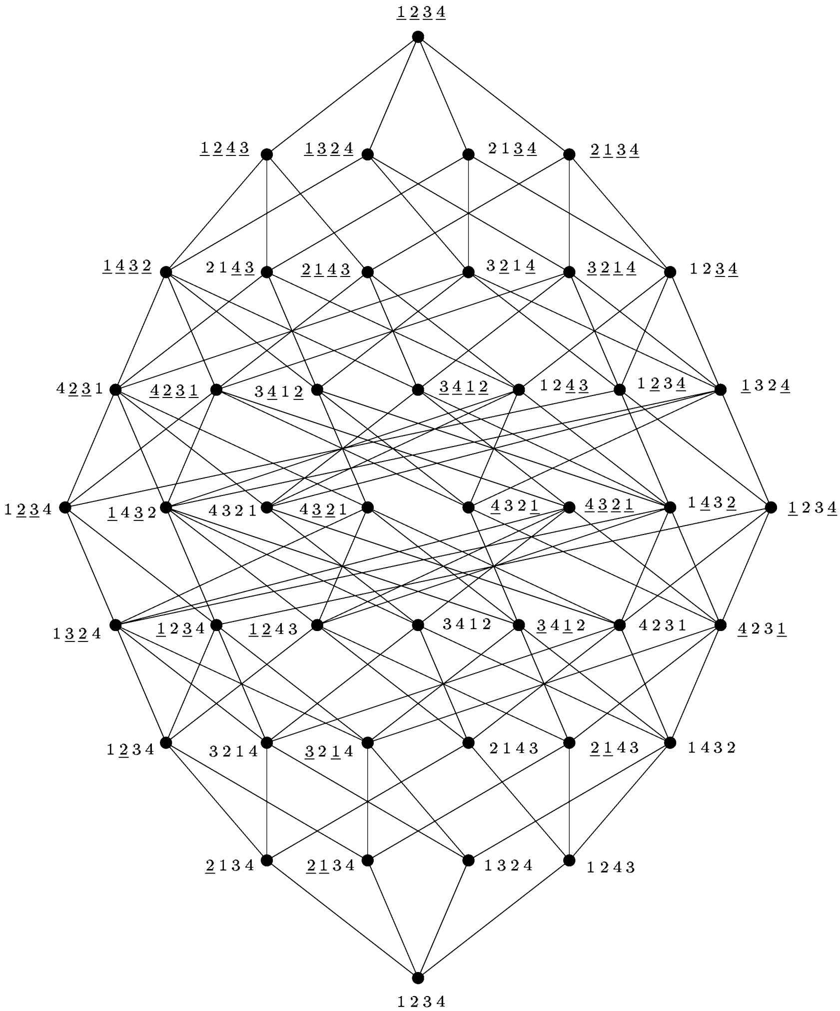


Fig. 9. $Invol(D_4)$.

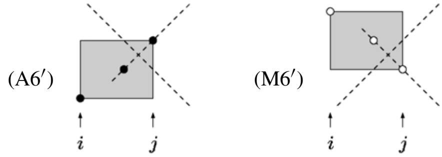


Fig. 10. Covering relation in $Invol(D_n)$: new cases.

Definition 6.1. Let $\sigma, \tau \in Invol(D_n)$. We say that (σ, τ) is a *good pair* in $Invol(D_n)$ if there exists a rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$, either non-central or central non-symmetric, such that the same conditions as in Definition 5.1 are satisfied, with the exceptions, if R is central non-symmetric, that:

(1) in cases $(A6, M1)$ and $(A6, M3)$, picture $A6$ is replaced by picture $A6'$, and in cases $(A1, M6)$ and $(A3, M6)$, picture $M6$ is replaced by picture $M6'$, as shown in Fig. 10;
(2) in the remaining cases, $(A3, M4)$, $(A4, M3)$, $(A4, M4)$, $(A4, M6)$, $(A6, M4)$, the presence in R of one more dot either of σ or of τ, which is in the orbit of one of those indicated in the pictures, is allowed.

Let (σ, τ) be a good pair in $Invol(D_n)$. The *main rectangle* of (σ, τ), denoted by $R(\sigma, \tau)$ is the rectangle $R = [i, j] \times [\sigma(i), \tau(i)]$ mentioned in Definition 6.1.

Table 12
Covering relation in $InvolD_n$: central cases

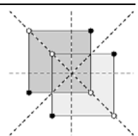
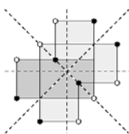
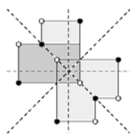
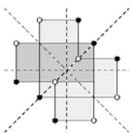
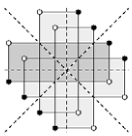
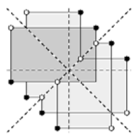
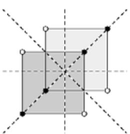
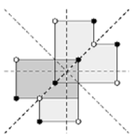
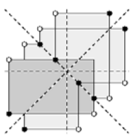
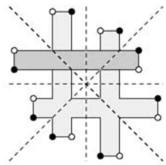
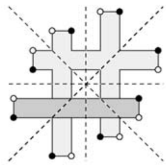
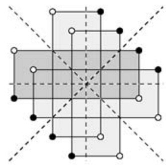
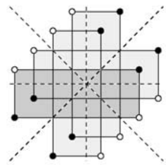
	M1	M2	M3	M4	M5	M6
A1	–	–	–	–	–	
A2	–	–	–	–	–	–
A3	–	–	–		–	
A4	–	–			–	
A5	–	–	–	–	–	–
A6		–			–	–

Table 13
The only case in which there is more than one possibility, i.e. $(A4, M4)$

$(A4, M4)a$	$(A4, M4)b$	$(A4, M4)c$	$(A4, M4)d$
			

Note that non-central cases are exactly the same as in $Invol(B_n)$, so they are described in Table 1. Central cases are new, and they are described in Table 12. The notation is the same used in Table 1.

The only case in which there is more than one possibility is $(A4, M4)$, as we show in Table 13. We set

$$H_{Invol(D_n)} = \{(\sigma, \tau) \in Invol(D_n)^2\colon\ (\sigma, \tau) \text{ is a good pair}\},$$

and define the *standard labeling* λ of $Invol(D_n)$ by associating, with every good pair $(\sigma, \tau) \in H_{Invol(D_n)}$, the pair $(i, j) \in [\pm n]^2$, if $R = [i, j] \times [\sigma(i), \tau(i)]$ is the main rectangle of (σ, τ).

By Theorem 2.15, $H_{Invol(D_n)}$ is a successor system and, since τ is uniquely determined by σ and by the label (i, j), λ is a good labeling.

Given $\sigma \in Invol(D_n)$, a pair $(i, j) \in [\pm n]^2$ is a *suitable label* of σ if there exists $\tau \in Invol(D_n)$, with $\lambda(\sigma, \tau) = (i, j)$. Such a τ, obviously unique, is called the *transformation* of σ with respect to (i, j) and it is denoted by

$$t_{(i,j)}^{Invol(D_n)}(\sigma).$$

Since an element of $Invol(D_n)$ is also in $Invol(B_n)$, we can consider the same types as defined in 5.2.

Definition 6.2. Let $\sigma, \tau \in Invol(D_n)$. The D-*type* of the pair (σ, τ) is

$$D\text{-}type(\sigma, \tau) = Dh.k,$$

if the B-type of (σ, τ) is $Bh.k$, referring to the cases as in Tables 3 to 10.

Note that in $Invol(D_n)$ some B-types cannot occur.

Proposition 6.3. *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. The B-type of (σ, τ) cannot be $B1.1$, $B1.4$ (see Table 4) or $B6.1$ (see Table 9).*

Proof. If B-$type(\sigma, \tau) = B1.1$ or $B1.4$, then $(di + 1, \sigma(ci))$ is a non-valid D-cell of (σ, τ), contradicting $\sigma < \tau$.

Now let $B\text{-}type(\sigma,\tau) = B6.1$. We may assume, without loss of generality, that $di = -n$. We have $ci = -1$ and $\sigma[-2,2] = n-2$, which implies $\tau[-2,2] = n-2$. So $neg(\sigma) = n-1$ and $neg(\tau) = n$, contradicting $\sigma, \tau \in D_n$. $\square$

We want to define the D-covering index Dci and the involution $\chi_{Invol(D_n)}(\sigma,\tau)$, which are the analogs of Bci and $\chi_{Invol(B_n)}(\sigma,\tau)$ in $Invol(B_n)$. In almost all cases the behavior in $Invol(D_n)$ is exactly the same as in $Invol(B_n)$, that is, $Dci = Bci$ and $\chi_{Invol(D_n)}(\sigma,\tau) = \chi_{Invol(B_n)}(\sigma,\tau)$. Only in some cases the approach in $Invol(D_n)$ is different with respect to $Invol(B_n)$. These are the cases represented in Tables 14 to 22.

In particular we consider the cases in which (σ,τ) is a D-special pair, in the sense of Definition 2.22, corresponding to cases 9(a), 9(b) and 10 of Fig. 5. This can occur in D-types $D3.7$, $D3.8$, $D3.10$, $D3.16$, $D3.17$.

Table 14
D-type D1.3

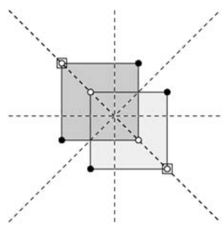
D1.3	
 D1.3	

Table 15
D-type D2

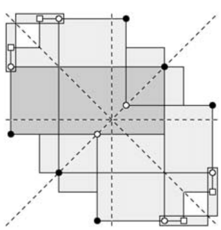
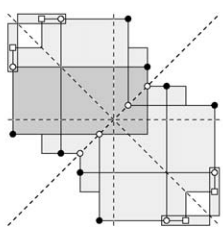
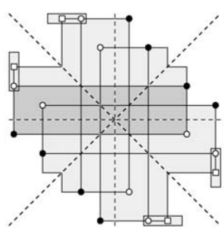
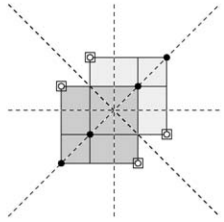
D2		
 D2.1	 D2.2	 D2.3

Table 16
D-type D3.7

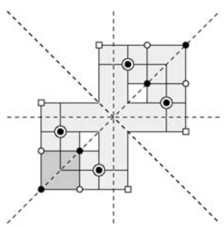
D3.7	



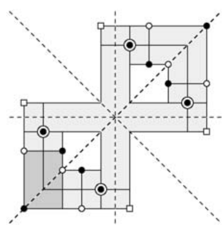
D3.7b

Table 17
D-type D3.8

D3.8	



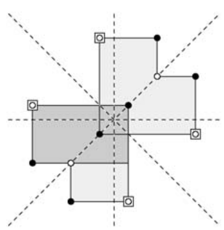
D3.8b



D3.8c

Table 18
D-type D3.10

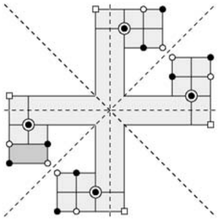
D3.10	



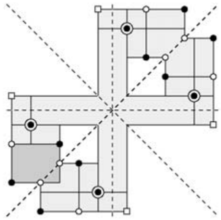
D3.10b

Table 19
D-type D3.16

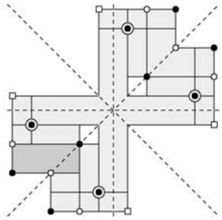
D3.16	



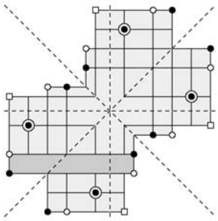
D3.16b



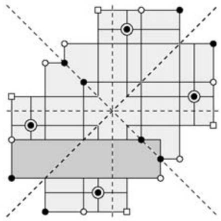
D3.16c



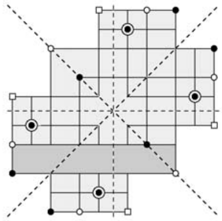
D3.16d



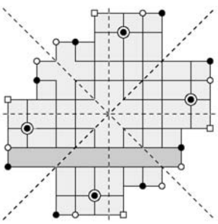
D3.16e



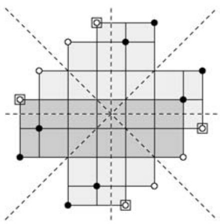
D3.16f



D3.16g



D3.16h



D3.16i

Definition 6.4. Let $\sigma, \tau \in \mathit{Invol}(D_n)$, with $\sigma < \tau$. The *D-covering index* of σ with respect to τ, denoted by $Dci_\tau(\sigma)$ (or simply Dci), is the minimal index $j \in [di + 1, n]$ such that there exists $\chi \in \mathit{Invol}(D_n)$, with

(1) $(\sigma, \chi) \in H_{\mathit{Invol}(D_n)}$;
(2) $\lambda(\sigma, \chi) = (di, j)$;
(3) $\chi(di) \leqslant \tau(di)$.

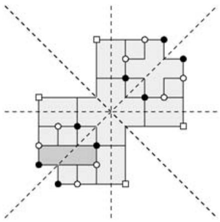
As it comes out from the pictures, Dci is always well defined.

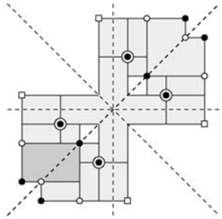
By definition, (di, Dci) is a suitable label of σ, so we can consider the involution

$$\chi_{\mathit{Invol}(D_n)}(\sigma, \tau) = t^{\mathit{Invol}(D_n)}_{(di, Dci)}(\sigma).$$

Table 20
D-type D3.17

D3.17


D3.17b


D3.17c

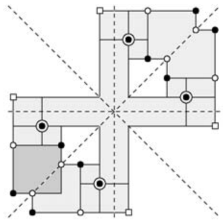

D3.17d

Table 21
D-type D5.3

D5.3

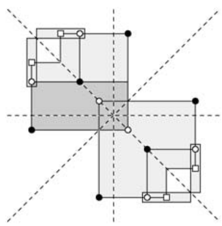
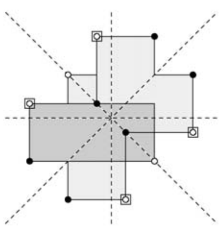
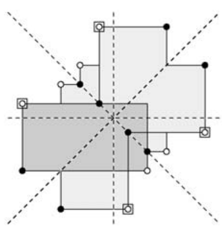
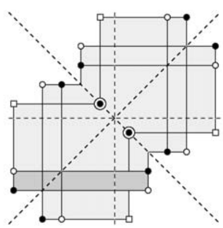

D5.3

Table 22
D-type D6.2

D6.2


D6.2a


D6.2b


D6.2c

In the pictures the involutions σ, τ and $\chi = \chi_{Invol(D_n)}(\sigma, \tau)$ are represented, with the same notation used in Tables 3 to 10.

In order to prove that the insertion property holds, we need the following definitions.

Definition 6.5. Let (σ, τ) be a good pair in $Invol(D_n)$. The *anti-D-multiplicity* of (σ, τ), denoted by $a\text{-}D\text{-}mult(\sigma, \tau)$, is

$$a\text{-}D\text{-}mult(\sigma,\tau) = \begin{cases} 0, & \text{if } a\text{-}case(\sigma,\tau) = A1, \\ 1, & \text{if } a\text{-}case(\sigma,\tau) = A2, A3, A4, A5, A6', \\ 2, & \text{if } a\text{-}case(\sigma,\tau) = A6. \end{cases}$$

Similarly, the *main D-multiplicity* of (σ, τ), denoted by $m\text{-}D\text{-}mult(\sigma, \tau)$, is

$$m\text{-}D\text{-}mult(\sigma,\tau) = \begin{cases} 0, & \text{if } m\text{-}case(\sigma,\tau) = M1, \\ 1, & \text{if } m\text{-}case(\sigma,\tau) = M2, M3, M4, M5, M6', \\ 2, & \text{if } m\text{-}case(\sigma,\tau) = M6. \end{cases}$$

The *D-multiplicity* of (σ, τ), denoted by $D\text{-}mult(\sigma, \tau)$, is

$$D\text{-}mult(\sigma,\tau) = a\text{-}D\text{-}mult(\sigma,\tau) + m\text{-}D\text{-}mult(\sigma,\tau).$$

The following definitions are valid in general for even-signed permutations (not necessarily involutions).

If $\sigma, \tau \in D_n$, with $\sigma < \tau$, are such that (σ, τ) is a B-exceptional pair (see Definition 5.4), then the *second covering transformation* in D_n of σ with respect to τ, denoted by $sct_\tau^{D_n}(\sigma)$, is defined in the same way as in B_n:

$$sct_\tau^{D_n}(\sigma) = sct_\tau^{B_n}(\sigma) = \sigma(di, sci)(-di, -sci).$$

The following result is the analog in D_n of Proposition 5.6.

Proposition 6.6. *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$, be such that (σ, τ) is a B-exceptional pair. Then*

$$sct_\tau^{D_n}(\sigma) \leqslant \tau.$$

Proof. Let $\chi = sct_\tau^{D_n}(\sigma)$. By Proposition 5.6, we have that (χ, τ) satisfies the B-condition. On the other hand, in every case the pair (χ, τ) has no new D-cells with respect to the pair (σ, τ). So (χ, τ), as well as (σ, τ), satisfies the D-condition. $\square$

In D_n there is one more *exceptional* case to consider.

Definition 6.7. Let $\sigma, \tau \in D_n$. Suppose that sci exists and that the set

$$\big\{ j \in [sci+1, n]\colon\ \sigma(j) \in \big[\sigma(di)+1, \sigma(sci)-1\big]\big\}$$

is not empty. Then the *third covering index* of σ with respect to τ, denoted by $tci_\tau(\sigma)$ (or simply tci), is

$$tci_\tau(\sigma) = \min\big\{ j \in [sci+1, n]\colon\ \sigma(j) \in \big[\sigma(di)+1, \sigma(sci)-1\big]\big\}.$$

Definition 6.8. Let $\sigma, \tau \in D_n$, with $\sigma < \tau$. We say that (σ, τ) is a *D-exceptional pair* if

(1) $\tau(di) = \sigma(ci)$;
(2) sci exists;
(3) $sci = -ci$;
(4) tci exists;
(5) $\sigma(tci) < -tci$.

An example of D-exceptional pair is $(\sigma, \tau) \in Invol(D_n)^2$, with $\sigma < \tau$, whose D-type is $D6.2c$ (see Table 22).

Definition 6.9. Let $\sigma, \tau \in D_n$, with $\sigma < \tau$, be such that (σ, τ) is a D-exceptional pair. The *third covering transformation* in D_n of σ with respect to τ, denoted by $tct_\tau^{D_n}(\sigma)$, is

$$tct_\tau^{D_n}(\sigma) = \sigma(di, tci)(-di, -tci).$$

Proposition 6.10. *Let $\sigma, \tau \in D_n$, with $\sigma < \tau$, be such that (σ, τ) is a D-exceptional pair. Then*

$$tct_\tau^{D_n}(\sigma) \leqslant \tau.$$

Proof. By the symmetry of the diagram and by Proposition 2.7, it suffices to prove that

$$\sigma(di, tci) \leqslant \tau.$$

Let $\chi = \sigma(di, tci)$. In order to prove that (χ, τ) satisfies the B-condition, we may assume, without loss of generality, that $di = -n$. Consider the rectangle $R = [di, tci - 1] \times [\sigma(di) + 1, \sigma(tci)]$. For every $(h, k) \in [\pm n]^2$, we have

$$\chi[h, k] = \begin{cases} \sigma[h, k] + 1, & \text{if } (h, k) \in R, \\ \sigma[h, k], & \text{if } (h, k) \notin R. \end{cases}$$

So we have to show that $\tau[h, k] \geqslant \sigma[h, k] + 1$, for every $(h, k) \in R$. Let $(h, k) \in R$. If $h < sci$, then the proof is the same as in Proposition 5.6. Now let $h \in [sci, tci - 1]$. Since $\sigma, \tau \in D_n$, $\sigma[-\tau(di), \tau(di)]$ and $\tau[-\tau(di), \tau(di)]$ have the same parity, so we have

$$\sigma\big[-\tau(di) - 1, \tau(di) + 1\big] \leqslant \tau\big[-\tau(di) - 1, \tau(di) + 1\big] + 1.$$

Then

$$\begin{aligned} \sigma[h, k] &= \sigma\big[-\tau(di) - 1, \tau(di) + 1\big] + 2 \\ &\leqslant \tau\big[-\tau(di) - 1, \tau(di) + 1\big] + 3 \\ &\leqslant \big(\tau[h, k] - 2\big) + 3. \end{aligned}$$

Thus $\tau[h, k] \geqslant \sigma[h, k] + 1$.

Finally, the pair (χ, τ) has no new D-cells with respect to the pair (σ, τ). So (χ, τ), as well as (σ, τ), satisfies the D-condition. □

We introduce the following notation. Given $\sigma \in D_n$, we set

$$\sigma i = \sigma^{-1}.$$

Given $\sigma, \tau \in D_n$, with $\sigma < \tau$, we set

$$\sigma m = \begin{cases} sct_\tau^{D_n}(\sigma), & \text{if } (\sigma, \tau) \text{ is a } B\text{-exceptional pair,} \\ tct_\tau^{D_n}(\sigma), & \text{if } (\sigma, \tau) \text{ is a } D\text{-exceptional pair,} \\ mct_\tau^{D_n}(\sigma), & \text{otherwise.} \end{cases}$$

Proposition 6.11. *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$, and let*

$$\chi = \chi_{Invol(D_n)}(\sigma, \tau).$$

Then

$$\chi = \begin{cases} \sigma m, & \textit{if } D\textit{-mult}(\sigma, \chi) = 0, 1, \\ \sigma mim, & \textit{if } D\textit{-mult}(\sigma, \chi) = 2, \\ \sigma mimm, & \textit{if } D\textit{-mult}(\sigma, \chi) = 3, \\ \sigma mimmm, & \textit{if } D\textit{-mult}(\sigma, \chi) = 4. \end{cases}$$

Proof. It can be checked case by case, looking at the pictures of Tables 4 to 11 and those of Tables 15 to 22, and using the description of the minimal covering transformation in D_n.

For example, if $D\text{-}type(\sigma, \tau) = D6.2b$, then $case(\sigma, \chi) = (A6, M4)$, so $D\text{-}mult(\sigma, \chi) = 3$, and $\chi = \sigma mimm$ is illustrated in Fig. 11. $\square$

We can now prove the insertion property of $Invol(D_n)$.

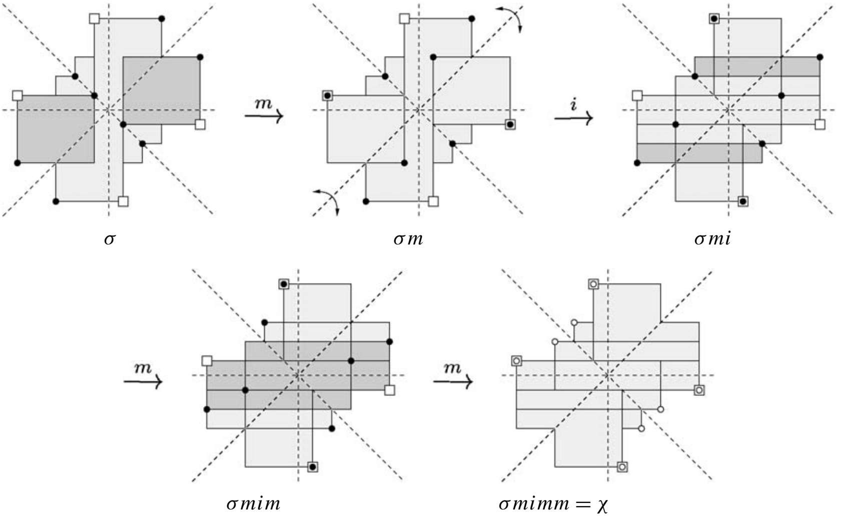


Fig. 11. Proof of Proposition 6.11.

Proposition 6.12. *Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. Then*

$$\chi_{Invol(D_n)}(\sigma, \tau) \leqslant \tau.$$

Proof. For every $\sigma \in D_n$, with $\sigma < \tau$, by Proposition 2.6, since τ is an involution, we have

$$\sigma i = \sigma^{-1} \leqslant \tau,$$

and by Propositions 2.4, 6.6 and 6.10, we have

$$\sigma m \leqslant \tau.$$

Thus $\chi_{Invol(D_n)}(\sigma, \tau) \leqslant \tau$ is a consequence of Proposition 6.11. □

The average between the length and the absolute length of $\sigma \in Invol(D_n)$ is

$$\rho(\sigma) = \frac{inv(\sigma) - neg(\sigma) + 2dna(\sigma)}{4}.$$

Proposition 6.13. *The pair $(H_{Invol(D_n)}, \rho)$ is a covering system of $Invol(D_n)$.*

Proof. By Proposition 6.12, $H_{Invol(D_n)}$ is an insertion system of $Invol(D_n)$. The ρ-base property is trivial. It remains to prove the ρ-increasing property. Consider $(\sigma, \tau) \in H_{Invol(D_n)}$. We have to prove that

$$\Delta\rho = \frac{\Delta inv - \Delta neg + 2\Delta dna}{4} = 1,$$

where $\Delta x = x(\tau) - x(\sigma)$.

In non-central cases (see Table 1) we have $\Delta neg = 0$, so $\Delta\rho$ is the same as in $Invol(B_n)$, and we already know that $\Delta\rho = 1$.

In central cases (see Table 12) we have

$$(\Delta inv, \Delta neg, \Delta dna) = \begin{cases} (4, 2, 1), & \text{in cases } (A1, M6'), (A6', M1), \\ (6, 2, 0), & \text{in cases } (A3, M4), (A4, M3), (A3, M6'), (A6', M3), \\ (8, 2, -1), & \text{in cases } (A4, M6), (A6, M4), \\ (8, 4, 0), & \text{in case } (A4, M4). \end{cases}$$

Thus in every case $\Delta\rho = 1$. □

We are now able to state and prove the gradedness of $Invol(D_n)$.

Theorem 6.14. *The poset $Invol(D_n)$ is graded, with rank function given by*

$$\rho(\sigma) = \frac{inv(\sigma) - neg(\sigma) + 2dna(\sigma)}{4},$$

for every $\sigma \in Invol(D_n)$. In particular $Invol(D_n)$ has rank

$$\rho\big(Invol(D_n)\big) = \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Proof. The first part is a consequence of Proposition 6.13. For the second part, note that the maximum w_0 of D_n, which is also the maximum of $Invol(D_n)$, is such that

$$\big(inv(w_0), neg(w_0), dna(w_0)\big) = \begin{cases} (n(2n-1), n, n), & \text{if } n \text{ is even,} \\ (n(2n-1)-1, n-1, n-1), & \text{if } n \text{ is odd.} \end{cases}$$

So

$$\rho\big(Invol(D_n)\big) = \rho(w_0) = \begin{cases} n^2/2, & \text{if } n \text{ is even,} \\ (n^2-1)/2, & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

We also have a characterization of the covering relation in $Invol(D_n)$: if $\sigma, \tau \in Invol(D_n)$, then $\sigma \lhd \tau$ if and only if (σ, τ) is a good pair in $Invol(D_n)$. And the transformation of σ with respect to a suitable label (i, j) actually is a *covering transformation*, denoted by

$$ct_{(i,j)}^{Invol(D_n)}(\sigma).$$

It remains to prove that $Invol(D_n)$ is *EL*-shellable and Eulerian. By the definitions of di and Dci, it follows that, given $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$, the *minimal label* of σ with respect to τ is (di, Dci), so

$$mct_{\tau}^{Invol(D_n)}(\sigma) = ct_{(di,Dci)}^{Invol(D_n)}(\sigma) = \chi_{Invol(D_n)}(\sigma, \tau).$$

Thus the *minimal chain* in $Invol(D_n)$ from σ to τ is the saturated chain

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd \cdots \lhd \sigma_k = \tau,$$

defined by

$$\sigma_i = mct_{\tau}^{Invol(D_n)}(\sigma_{i-1}), \tag{5}$$

for every $i \in [k]$.

In order to prove that $Invol(D_n)$ is *EL*-shellable, we prove the increasing and the decreasing properties.

Proposition 6.15 *(Increasing property). Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. The minimal chain*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd \cdots \lhd \sigma_k = \tau,$$

defined in (5) *has increasing labels.*

Proof. Suppose, by contradiction, that at a certain step there is a decrease in the labels. We may assume, without loss of generality, that this happens at the first step. So

$$\sigma \underset{(di,Dci)}{\lhd} \sigma_1 \underset{(i,j)}{\lhd} \sigma_2,$$

with $(i, j) < (di, Dci)$. So either $i < di$ or $i = di$ and $j < Dci$. If $i < di$, since σ and τ must differ at the index i, the minimality of di is contradicted. So suppose $i = di$ and $j < Dci$. Let $v = \sigma_2(di)$. Since $\sigma_2 \leqslant \tau$, we have $v \leqslant \tau(di)$. But $[di, j] \times [\sigma(di), v]$ is the main rectangle of (σ, χ), for some $\chi \in Invol(D_n)$. Thus the minimality of Dci is contradicted. □

Proposition 6.16 *(Decreasing property). Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$, and let*

$$\sigma = \sigma_0 \lhd \sigma_1 \lhd \cdots \lhd \sigma_k = \tau,$$

be the minimal chain defined in (5)*. Every saturated chain from σ to τ, different from the minimal one, say*

$$\sigma = \tau_0 \lhd \tau_1 \lhd \cdots \lhd \tau_k = \tau,$$

has at least one decrease in the labels.

Proof. The proof is essentially the same as that of [14, Theorem 4.4]. There are a few further cases to be considered, which can be handled with the same techniques, and they are left to the reader. □

Next theorem is an immediate consequence of the increasing and the decreasing properties.

Theorem 6.17. *The poset $Invol(D_n)$ is EL-shellable, having the standard labeling as an EL-labeling.*

We can now prove that the condition of Theorem 2.2 holds for the poset $Invol(D_n)$, and thus that it is Eulerian.

Theorem 6.18. *The poset $Invol(D_n)$ is Eulerian.*

Proof. Suppose we label the edges of the Hasse diagram of $Invol(D_n)$ with the standard labeling.

Let $\sigma, \tau \in Invol(D_n)$, with $\sigma < \tau$. By Theorem 2.2, we only have to show that there is exactly one saturated chain from σ to τ with decreasing labels.

Note that, if $\chi \in D_n$, then $w_0\chi$ not necessarily is in D_n. More precisely, since

$$neg(w_0\chi) = n - neg(\chi),$$

we have that $w_0\chi \in D_n$ if and only if n is even.

But, if $\chi \in D_n$, we can consider the signed permutation of B_{n+1}, which we denote by $ext(\chi)$, whose diagram is obtained from the diagram of χ, by adding rows and columns $\pm(n+1)$, and either the two dots in cells $(n+1, n+1)$ and $(-n-1, -n-1)$, if n is even, or the two dots in cells $(n+1, -n-1)$ and $(-n-1, n+1)$, if n is odd. In every case $w_0 ext(\chi) \in D_{n+1}$.

Consider the two even-signed permutations $w_0 ext(\sigma)$ and $w_0 ext(\tau)$. They are involutions of D_{n+1} and it's easy to see that

$$w_0 ext(\tau) < w_0 ext(\sigma).$$

Since the standard labeling of $Invol(D_{n+1})$ is an EL-labeling, there is exactly one saturated chain from $w_0 ext(\tau)$ to $w_0 ext(\sigma)$ with non decreasing labels (it actually has strictly increasing labels), and necessarily all the elements of this chain have the form $w_0 ext(\chi)$, for some $\chi \in Invol(D_n)$:

$$w_0 ext(\tau) = w_0 ext(\chi_0) \lhd w_0 ext(\chi_1) \lhd \cdots \lhd w_0 ext(\chi_k) = w_0 ext(\sigma).$$

Then

$$\sigma = \chi_k \lhd \cdots \lhd \chi_1 \lhd \chi_0 = \tau$$

is the unique saturated chain from σ to τ with decreasing labels. $\square$

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